

Grover's algorithm (Lecture 22)

Setup.

 $f: \{0, 1\}^n \rightarrow \{0, 1\}$ oracle function, such thatthere is unique $x^* \in \{0, 1\}^n$ s.t. $f(x^*) = 1$.we want to know x^* ("marked item")classically: we need $N = 2^n$ steps (check $f(x) \stackrel{?}{=} 1$ for all x)

Quantum algorithm

• start with $|0 \dots 0\rangle \in (\mathbb{C}^2)^{\otimes n}$

• apply the Hadamard gate on each qubit

$$|\psi_0\rangle = H^{\otimes n} |0 \dots 0\rangle = \frac{1}{\sqrt{2^n}} \sum_{x \in \{0, 1\}^n} |x\rangle$$

• apply the phase query $U_f: |x\rangle \mapsto (-1)^{f(x)} |x\rangle$

$$|\varphi_0\rangle = U_f |\psi_0\rangle$$

• then apply the Grover diffusion operator

repeat

$$D|x\rangle = \left(\frac{1}{2^{n-1}} \sum_{y \in \{0, 1\}^n} |y\rangle \right) - |x\rangle$$

$$|\psi_1\rangle = D|\varphi_0\rangle$$

• $k \sim 2^{n/2 - 2} \cdot \pi$; measure $|\psi_k\rangle$ in the basis $(|x\rangle)_{x \in \{0, 1\}^n}$ Prop 1. D is unitary

• matrix presentation of D :

$$D = \begin{bmatrix} \frac{2}{N} - 1 & \frac{2}{N} & & \\ \frac{2}{N} & \frac{2}{N} - 1 & & \\ & & \ddots & \\ & & & \frac{2}{N} - 1 \end{bmatrix}$$

$$\leadsto D^\dagger = D$$

• $D \left(\sum_x \alpha_x |x\rangle \right) = \sum_x \left(2 \cdot \frac{\sum_y \alpha_y}{N} - \alpha_x \right) |x\rangle$

cont.) i.e. coeff. of $|x\rangle$ is transformed as

$$d_x \mapsto 2\bar{d} - d_x \quad \bar{d} = \text{average of the } d_y$$

$$\mapsto D^2 = I_N$$

Why this works

$$- |\psi_0\rangle = \frac{1}{\sqrt{N}} \sum_x |x\rangle.$$

$$- |\varphi_0\rangle = \frac{1}{\sqrt{N}} \left(\left(\sum_{x \neq x^*} |x\rangle \right) - |x^*\rangle \right)$$

$$\text{average coeff} = \frac{1}{N} \cdot \left((N-1) \times \frac{1}{\sqrt{N}} + 1 \times \left(-\frac{1}{\sqrt{N}} \right) \right) = \frac{N-2}{N^{3/2}}$$

$$- |\psi_1\rangle = \sum_{x \neq x^*} \left(\frac{2(N-2)}{N^{3/2}} - \frac{1}{\sqrt{N}} \right) |x\rangle + \left(\frac{1}{\sqrt{N}} + \frac{N-2}{N^{3/2}} \right) |x^*\rangle$$

$$\frac{2}{\sqrt{N}} - \frac{1}{\sqrt{N}} = \frac{1}{\sqrt{N}} = \frac{4}{N^{3/2}} \quad \frac{2N-2}{N^{3/2}}$$

$$\dots - |\varphi_k\rangle = \sum_{x \neq x^*} \alpha_x^{(k)} |x\rangle + (-\alpha_{x^*}^{(k)}) |x^*\rangle, \quad \alpha_{x^*}^{(k)} > \alpha_x^{(k)} > 0$$

$$\text{average } \beta : 0 < \beta < \alpha^{(k)} \quad \text{so} \quad 2\beta - \alpha^{(k)} < \alpha^{(k)}$$

$\mapsto |\psi_k\rangle$ converges to $|x^*\rangle$

• Analyzing the running time (§ 22.1-3)

Note : $|\psi_k\rangle, |\varphi_k\rangle$ are linear comb. of

$|\psi_0\rangle = \frac{1}{\sqrt{N}} \sum_x |x\rangle$ and $|x^*\rangle$ ($|x\rangle$ for $x \neq x^*$ have the same coefficients).

(w) we can take the real 2-dim span of $|x^*\rangle$ and

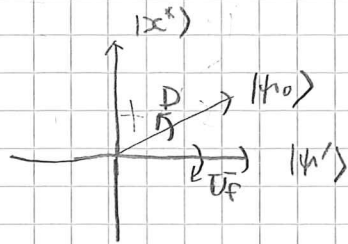
$$|\psi'\rangle = \frac{1}{\sqrt{N-1}} \sum_{x \neq x^*} |x\rangle$$

U_f : reflection along the span of $|\psi'\rangle$

D : reflection along the span of $|\psi_0\rangle$

$\leadsto D U_f$: rotation by 2θ ; where

$$|\psi_0\rangle = \sin\theta |x^*\rangle + \cos\theta |\psi'\rangle$$



$$\sin\theta = \frac{1}{\sqrt{N}}, \quad \cos\theta = \sqrt{\frac{N-1}{N}}$$

$$\theta \sim \frac{1}{\sqrt{N}} \quad \text{when } N \gg 1$$

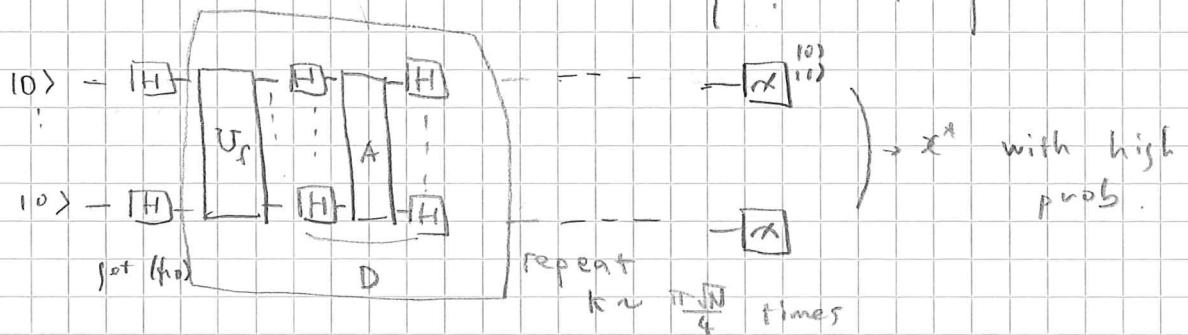
when $2k\theta \sim \frac{\pi}{2}$ we have $(D U_f)^k |\psi_0\rangle \sim |x^*\rangle$
 (i.e. $k \sim \frac{\pi\sqrt{N}}{4}$) $|\psi_k\rangle$.

Quantum circuit for G-algo. (22.1.1)

$H^{\otimes n}$ exchanges $|0\dots 0\rangle$ and $\frac{1}{\sqrt{N}} \sum_{x \in \{0,1\}^n} |x\rangle$
 fixed pt of D .

$$\text{so } D = H^{\otimes n} A H^{\otimes n}$$

$$A = \begin{bmatrix} -1 & 0 & \dots & 0 \\ 0 & 1 & & \\ & & \ddots & \\ 0 & & & 1 \end{bmatrix} \begin{matrix} |100\dots 0\rangle \\ \\ \\ |100\dots 0\rangle \end{matrix}$$



Multiple marked items (22.1.4)

Suppose there are K solutions to $f(x) = 1$.

$$|x_i^*\rangle \text{ s.t. } f(x_i^*) = 1$$

$\leadsto$ consider the same algorithm with U_f, D .

$$\text{so } |\psi_0\rangle = \frac{1}{\sqrt{N}} \left(\sum_{x \neq x_i^*} |x\rangle - \sum_{\text{osick}} |x_i^*\rangle \right), \quad |\psi_k\rangle = D |\psi_0\rangle, \dots$$

$\leadsto$ we'll be working with the 2-dim span of

$$|x^*\rangle = \frac{1}{\sqrt{K}} \sum_{\text{osick}} |x_i^*\rangle, \quad |\psi'\rangle = \frac{1}{\sqrt{N-K}} \sum_{x \neq x_i^*} |x\rangle$$

$D U_f$: rotation by 2θ , with

$$|\psi_0\rangle = \sin \theta |x^*\rangle + \cos \theta |\psi'\rangle$$

$$\text{and } \sin \theta = \sqrt{\frac{K}{N}}, \quad \cos \theta = \sqrt{\frac{N-K}{N}}$$

$\Rightarrow$ we should stop at the $k \sim \frac{\pi}{4} \cdot \sqrt{\frac{N}{K}}$ step.

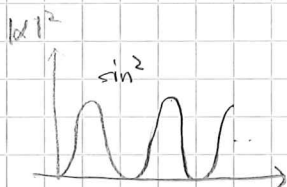
• When the number of marked points is unknown.

pick a random number k between 0 and $\frac{\pi \sqrt{N}}{4}$

measure $(D U_f)^k |\psi_0\rangle$ in the $|x\rangle_{x \in \{0, \dots, N-1\}}$ basis.

$$\text{Success probability} \sim \int_0^1 \sin^2 \left(\frac{\pi}{2} \sqrt{K} t \right) dt \sim \frac{1}{2}$$

$K = \# \text{ marked items}$



$$(D U_f)^k |\psi_0\rangle = \alpha |x^*\rangle + \beta |\psi'\rangle$$