

Error correction (lecture 27)

(like everything in real world) we have to allow some margin of errors in operations for quantum computation

→ can we detect and correct errors?

classical situation (§27.1)

- string of bits 0110 --

- after some manipulation, some bit might flip $0 \leftrightarrow 1$ by an error.

one solution: 3-bit repetition code

represent 0 by $\bar{0} = 000$, 1 by $\bar{1} = 111$.

and operate on blocks of 3-bits.

$\text{AND}(\bar{0}, \bar{1}) = \bar{0}$, $\text{OR}(\bar{0}, \bar{1}) = \bar{1}$ etc.

→ expect results of the form $\bar{0}\bar{1}\bar{0} \dots = 000111000 \dots$

error could introduce bit flips 000101001 --

to detect errors: take each block $x = x_0 x_1 x_2$

and compute "majority vote" $\text{MAJ}(x_0, x_1, x_2)$

010 → 0, etc.

von Neumann '56) if we use NOT, AND, OR gates

and the error occurs independently in the gates.

in a circuit of size n , we can make a more reliable circuit of size $O(n \log n)$

- quantum setting (§27.2)

New constraints / difficulties!

1. we have to work with states in a continuous param space: qubit $\in \mathbb{C}^2$ (or Bloch sphere...)

2. No-cloning theorem $\Rightarrow$ we cannot "duplicate" quantum states for naive generalization of classical scheme

3. Measurement will collapse the state and remove benefit of quantum computation

Model for errors:

Setting: N -qubit system $(\mathbb{C}^2)^{\otimes N}$.

errors happen at each qubit

$$|\psi\rangle \mapsto (I_{2^k} \otimes U \otimes I_{2^{N-k-1}}) |\psi\rangle \quad (U \in U(2))$$

at the k -th qubit

Prop. For a fixed $|\psi\rangle$, the vectors

$(I_{2^k} \otimes U \otimes I_{2^{N-k-1}}) |\psi\rangle$ belongs to a 4-dim space.

Proof. $U(2) \subset M_2(\mathbb{C})$, $M_2(\mathbb{C})$ is a 4-dim vec. sp.

with basis $\begin{bmatrix} 1 & \\ & 1 \end{bmatrix}, \begin{bmatrix} & 1 \\ 1 & \end{bmatrix}, \begin{bmatrix} 1 & \\ & -1 \end{bmatrix}, \begin{bmatrix} & 1 \\ 1 & -1 \end{bmatrix}$

Ex. Given $|\psi_0\rangle = \alpha|0\rangle \otimes |v\rangle + \beta|1\rangle \otimes |w\rangle \in \mathbb{C}^2 \otimes (\mathbb{C}^2)^{\otimes N-1}$

An error on the 0-th qubit $(U \otimes I_{2^{N-1}}) |\psi_0\rangle$

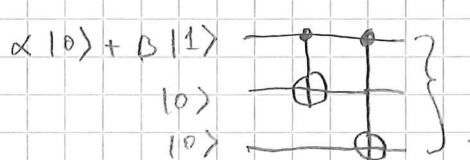
$$|\psi_1\rangle = \alpha|1\rangle \otimes |v\rangle + \beta|0\rangle \otimes |w\rangle, \quad |\psi_2\rangle = \alpha|0\rangle \otimes |v\rangle - \beta|1\rangle \otimes |w\rangle$$

$$|\psi_3\rangle = \alpha|1\rangle \otimes |v\rangle - \beta|0\rangle \otimes |w\rangle$$

To protect against "bit flip" $|\psi_0\rangle \mapsto |\psi_1\rangle$

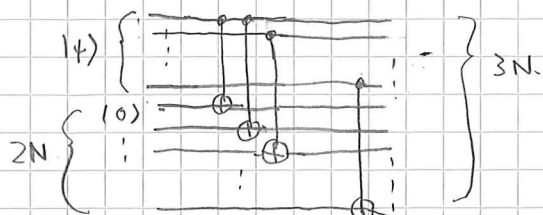
We can imitate 3-bit repetition

$$\alpha|0\rangle + \beta|1\rangle \mapsto \alpha|000\rangle + \beta|111\rangle \quad (\text{on each qubit})$$

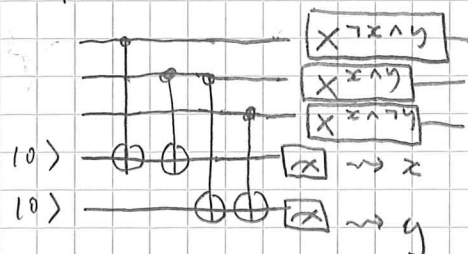


this is not cloning, but
some unitary gate on
 $|\psi\rangle \otimes |00\rangle$

So overall it will look like



To perform error correction on such coding



$X^{zx \wedge y}$: X gate if $x=0$ and $y=1$

$X^{xy \wedge y}$: X gate if $x=1$ and $y=1$

How this works : Suppose we expect $\alpha|1000\rangle + \beta|1111\rangle$
if the bit flip happened in the 1-th qubit
so we have $\alpha|1010\rangle + \beta|1101\rangle$

first CNOT gives $\alpha|101000\rangle + \beta|101100\rangle \rightarrow$
 $\alpha|101000\rangle + \beta|110110\rangle$

second CNOT gives $\alpha|101000\rangle + \beta|110110\rangle \rightarrow \alpha|101010\rangle + \beta|110110\rangle$
 $\sim \alpha|101011\rangle + \beta|110111\rangle$ by all CNOTs

measurement gives $x=1, y=1$ with prob. $\frac{1}{2}$

$X^?$ gives $X^0 \otimes X^1 \otimes X^0 = I_2 \otimes X \otimes I_2$

$\alpha|1010\rangle + \beta|1101\rangle \rightarrow \alpha|1000\rangle + \beta|1111\rangle$

In general, error will give something like

$$(I_2 \otimes R_\theta \otimes I_2)|\psi\rangle = \alpha (\cos\theta |1000\rangle + \sin\theta |1010\rangle) + \beta e^{2i\theta} (\cos\theta |1111\rangle - \sin\theta |1101\rangle)$$

after CNOTs

$$\cos\theta (\alpha |100000\rangle + \beta e^{2i\theta} |111100\rangle) + \sin\theta (\alpha |101011\rangle - \beta e^{2i\theta} |110111\rangle)$$

after observing $(x, y) = (0, 0)$ or $(1, 1)$,

$X^{xy \wedge y}$ will give $\alpha|1000\rangle \pm \beta e^{2i\theta} |1111\rangle$

cont.) but this won't help against phase flip
error $\alpha|000\rangle + \beta|111\rangle \rightarrow \alpha|000\rangle - \beta|111\rangle$