

Prob 4

1) $A^\dagger = A$ for $A \in M_2(\mathbb{C})$ means

$$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad a, d \text{ real}, \quad b = c^\dagger$$

$$\text{Tr } A = 0 \text{ means } d = -a$$

$$\text{so } A = \begin{bmatrix} a & c^\dagger \\ c & -a \end{bmatrix} = a \begin{bmatrix} 1 & z \\ \bar{z} & -1 \end{bmatrix} + s \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} + t \begin{bmatrix} 0 & i \\ i & 0 \end{bmatrix}$$

$a, s, t \text{ real} \quad (c = s + it)$

2) $(U A U^\dagger) = U^\dagger{}^\dagger A^\dagger U^\dagger = U A U^\dagger$

$$\text{Tr}(U A U^\dagger) = \text{Tr}(\underbrace{U^\dagger U}_I A) = \text{Tr}(A)$$

3) $\text{Ad}_{\exp(itZ)}(A) = e^{itZ} A e^{-itZ}$

from $Z^\dagger = Z$, $(e^{itZ})^\dagger = e^{-itZ^\dagger}$

Differential for t is $iZ e^{itZ} A e^{-itZ} - (e^{itZ})^\dagger iZ$

$$= [Z, \text{Ad}_{\exp(itZ)}(A)]$$

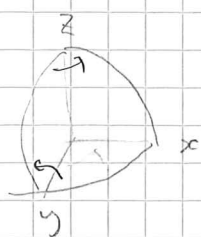
so $\text{Ad}_{\exp(itZ)}(A)$ is the solution to eq.

$$\frac{d B_t}{dt} = [iZ, B_t], \quad B_0 = A.$$

$$[iZ, X] = -2Y, \quad [iZ, Y] = 2X, \quad [iZ, Z] = 0$$

this is the eq. for rot. around Z -axis.

4) $\text{Ad}_{\exp(iY)}$ is the rot. around Y -axis.



these two generate a subgroup of $\text{SO}(3) = \{ \text{rot. around origin in 3D sp} \}$

of $\dim \geq 2$

so should be eq. to $\text{SO}(3)$

Prob 2

• $U(2) / \{ e^{i\alpha} I_2 \}$ is isom to $SO(3)$.

$A \mapsto \text{Ad}_U(A)$ on $\mathfrak{E}$ is trivial

iff $U = e^{i\alpha} I_2$.

• rot. in 3D-sp can be written as

(rot around z , angle β) • (rot ar. y , ang γ)

= (rot ar z , ang δ')

this gives $W = e^{i\alpha} e^{i\beta Z} e^{i\gamma Y} e^{i\delta Z}$

($2\pi = \beta', \dots$)

(N.C. Thm 4.1)

Finding $W = e^{i\alpha} A X B X C$, $ABC = I_2$

Set $A = e^{i\beta Z} e^{i\frac{\gamma}{2} Y}$, $B = e^{-i\frac{\gamma}{2} Y} e^{-i\frac{\delta+\beta}{2} Z}$

$C = e^{i\frac{\delta-\beta}{2} Z}$

($\Rightarrow ABC = e^{i(\beta + (-\frac{\delta+\beta}{2}) + \frac{\delta-\beta}{2})Z} = e^{i0 \cdot Z} = I_2$)

Claim $X B X = e^{i\frac{\gamma}{2} Y} e^{i\frac{\delta+\beta}{2} Z}$.

left hand side: $\text{Ad}_X(B)$

$\text{Ad}_X(Y) = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} = -Y$

$\text{Ad}_X(Z) = -Z$.

so $\text{Ad}_X(e^{-i\frac{\gamma}{2} Y}) = \exp(\text{Ad}_X(-i\frac{\gamma}{2} Y)) = e^{i\frac{\gamma}{2} Y}$

By claim, $A X B X C = e^{i\beta Z} e^{i\frac{\gamma}{2} Y} e^{i\frac{\gamma}{2} Y} e^{i\frac{\delta+\beta}{2} Z} e^{i\frac{\delta-\beta}{2} Z}$
 $= e^{i\beta Z} e^{i\gamma Y} e^{i\delta Z}$