MAT3420 2024, Exercises week 5, to be discussed Tuesday 30 January

Exercise 1. Let $\mathcal{H} = \mathbb{C}^2$ with its computational basis $\{|0\rangle, |1\rangle\}$. Compute the norm $\| |\psi\rangle \|$ of the vector

$$|\psi\rangle = \frac{2+3i}{5}|0\rangle + \frac{2\sqrt{3}i}{5}|1\rangle .$$

Exercise 2. (This exercise verifies that a unitary operator on a finite-dimensional Hilbert space preserves inner products using the matrix representation of U in the computational basis.)

(a) Let $\mathcal{H} = \mathbb{C}^2$ with its computational basis $\{|0\rangle, |1\rangle\}$. Suppose that U is a unitary operator on $\mathcal{H}$ and denote $\hat{U}$ its matrix with respect to the computational basis as given in [1, Theorem 2.3.2], so

$$U = \sum_{j,k=0,1} \hat{U}_{j,k} |j\rangle \langle k| .$$

Let $\hat{U}^\dagger$ be the Hermitian adjoint of the matrix $\hat{U}$ and I the identity 2×2 matrix. With matrix notation, we obtain that $\hat{U}^\dagger \hat{U} = I = \hat{U}\hat{U}^\dagger$. (These two identities express the fact that the columns of $\hat{U}$ form an orthonormal set in $\mathbb{C}^2$, which is shown by computing all four inner product of the vectors $U(|0\rangle)$ and $U(|1\rangle)$ using the definition of adjoint (equation (2.3.11) in [1]).)

Let $|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$ be a vector in $\mathbb{C}^2$ and let

$$\hat{\psi} := \begin{bmatrix} \alpha_0 \\ \alpha_1 \end{bmatrix}$$

Verify that the matrix product $\hat{U}\hat{\psi}$ is the matrix (2×1 column vector) $\hat{U\psi}$ corresponding to $U(|\psi\rangle)$. Show that

$$(\hat{U\psi})^\dagger = (\hat{\psi})^\dagger (\hat{U})^\dagger,$$

where we have matrix multiplication in the right-hand side (this is similar to the result about the transpose of a product $(AB)^T = B^T A^T$ of matrices). Conclude that

$$(U|\psi\rangle)^* (U|\phi\rangle) = \langle\psi|\phi\rangle,$$

for any two vectors $|\psi\rangle, |\phi\rangle$ in $\mathcal{H}$. (In other words, U preserves the inner product of arbitrary vectors). Note that the notation $|\chi\rangle^*$ means the adjoint vector of $|\chi\rangle$ for a vector in $\mathcal{H}$, and is computed in the chosen basis as the Hermitian conjugate $(\hat{\chi})^\dagger$ of the column $\hat{\chi}$. A short-hand notation then will be $|\chi\rangle^\dagger$.

(b) Write an alternative argument of the fact that U preserves inner products in $\mathcal{H}$ using the existence of adjoint of U in [1, Definition 2.3.4] and the properties of an inner product (without the need to choose an orthonormal basis).

(c) Note: The fact that unitary operators preserve inner products is valid in an arbitrary, not necessarily finite dimensional Hilbert space, and can be proved using the existence of adjoint.

Exercise 3. With the notation of example 2.4.4 in [1], verify that the matrix P is unitary. This is the Hadamard matrix (in dimension 2.)

Exercise 4. (This is identity (2.8.1) in [1]) Let $\mathcal{H} = \mathbb{C}^2$ with its computational basis $\{|0\rangle, |1\rangle\}$ and let $|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$, $|\phi\rangle = \beta_0 |0\rangle + \beta_1 |1\rangle$ be vectors in $\mathcal{H}$. Identifying $|\psi\rangle$ with its 2×1 matrix $\hat{\psi}$ as in Exercise 2, and similarly for $|\phi\rangle$, show that

$$(|\psi\rangle \otimes |\phi\rangle)^\dagger = \langle\psi| \otimes \langle\phi| .$$

Hint: use Equations (2.6.7), (2.6.8) and (2.6.10) in [1].

Exercise 5. Verify identity (2.8.4) in [1]. Hint: suppose $i, j, k, l \in \{0, 1\}$.

Exercise 6. Do the computations that show how to view the 8 elements in the computational basis for $\mathbb{C}^8$ as tensor product vectors of the form $|i\rangle \otimes |j\rangle \otimes |k\rangle$, or, in compressed form $|ijk\rangle$, where $i, j, k \in \{0, 1\}$.

Exercise 7. Exercise 3.3.1 in [1].

Exercise 8. (Postponned to week 7.) Consider a 2-qubit state (a unit vector in a composite system $\mathbb{C}^2 \otimes \mathbb{C}^2$) given in the form

$$|\varphi\rangle = \gamma_{00} |00\rangle + \gamma_{01} |01\rangle + \gamma_{10} |10\rangle + \gamma_{11} |11\rangle$$

for complex scalars $\gamma_{00}, \gamma_{01}, \gamma_{10}, \gamma_{11}$ satisfying $|\gamma_{00}|^2 + |\gamma_{01}|^2 + |\gamma_{10}|^2 + |\gamma_{11}|^2 = 1$. Prove that $|\varphi\rangle$ is a product state in the form

$$(\alpha_0 |0\rangle + \alpha_1 |1\rangle) \otimes (\beta_0 |0\rangle + \beta_1 |1\rangle)$$

if and only if $\gamma_{00}\gamma_{11} = \gamma_{01}\gamma_{10}$. Hint: to prove the condition is sufficient, try to express the γ's as products of α_j's and β_k's for suitable choices, and use that you may multiply a complex number by $e^{i\theta}$ without changing its modulus, where i is the imaginary complex number in $\mathbb{C}$ and θ is an angle in $[0, 2\pi)$.

References

[1] P. Kaye, R. Laflamme and M. Mosca, An Introduction to Quantum Computing, Oxford University Press, 2007.