29th February, 2024

COURSE MAT3420

Mandatory assignment 1 of 1

Submission deadline

Thursday 14th MARCH 2024, 14:30 in Canvas (canvas.uio.no).

Instructions

Note that you have **one attempt** to pass the assignment. This means that there are no second attempts.

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file. Scanned pages must be clearly legible. The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. Remember to include all relevant plots and figures. All aids, including collaboration, are allowed, but the submission must be written by you and reflect your understanding of the subject. If we doubt that you have understood the content you have handed in, we may request that you give an oral account.

In exercises where you are asked to write a computer program, you need to hand in the code along with the rest of the assignment. It is important that the submitted program contains a trial run, so that it is easy to see the result of the code.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) no later than the same day as the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

To pass the assignment you need to solve at least three of the six subproblems in this set. You can freely choose what subproblems to solve, but remember to justify your solution.

Problem 1. For a real number $\theta \in [0, \pi)$ there is a rotation unitary matrix defined by

$$R_y(2\theta) = \begin{bmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{bmatrix}.$$

Let $\{|0\rangle, |1\rangle\}$ be the computational basis of a 1-qubit system $\mathcal{H}_A$, and consider the new orthonormal basis $\{R_y(2\theta)|0\rangle, R_y(2\theta)|1\rangle\}$ for $\mathcal{H}_A$. Denote

$$|\theta\rangle := R_y(2\theta)|0\rangle = \cos(\theta)|0\rangle + \sin(\theta)|1\rangle. \tag{1}$$

By known trigonometric identities $\cos(\theta+\pi/2) = -\sin(\theta)$ and $\sin(\theta+\pi/2) = \cos(\theta)$, we then have

$$R_y(2\theta)|1\rangle = -\sin(\theta)|0\rangle + \cos(\theta)|1\rangle = \left|\theta + \frac{\pi}{2}\right\rangle. \tag{2}$$

(a) Write down the rule of a von Neumann measurement performed on system $\mathcal{H}_A$ from the state $|\varphi\rangle = |0\rangle$ with respect to the basis $\{|\theta\rangle, \left|\theta + \frac{\pi}{2}\right\rangle\}$. Answer the same question for the state $|\varphi\rangle = |1\rangle$.

Suppose we have a 2-qubit bipartite system $\mathcal{H}_A \otimes \mathcal{H}_B$ that is given in the Bell state

$$|\beta_{00}\rangle = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle).$$

Define projection operators on $\mathcal{H}_A \otimes \mathcal{H}_B$ by

$$P_\theta = |\theta\rangle\langle\theta| \otimes I_{\mathcal{H}_\mathcal{B}},\ P_{\theta+\frac{\pi}{2}} = \left|\theta + \frac{\pi}{2}\right\rangle\left\langle\theta + \frac{\pi}{2}\right| \otimes I_{\mathcal{H}_\mathcal{B}}.$$

(b) Show that $\langle 00|P_\theta|00\rangle = \cos^2(\theta)$, $\langle 11|P_\theta|11\rangle = \sin^2(\theta)$. Find also the values $\langle 00|P_\theta|11\rangle$ and $\langle 11|P_\theta|00\rangle$. Write down the similar formulas with $P_{\theta+\frac{\pi}{2}}$ instead of P_θ.

(c) Suppose that a projective measurement is performed on the system $\mathcal{H}_A \otimes \mathcal{H}_B$ from the state $|\beta_{00}\rangle$ with respect to the decomposition $I = P_\theta + P_{\theta+\frac{\pi}{2}}$. Show that the measurement outputs outcome "θ" with probability $1/2$, leaving the system in the state

$$\cos(\theta)|\theta\rangle|0\rangle + \sin(\theta)|\theta\rangle|1\rangle = |\theta\rangle|\theta\rangle,$$

and outputs outcome "$\theta + \frac{\pi}{2}$" with probability $1/2$, leaving the system in the state

$$-\sin(\theta)\left|\theta + \frac{\pi}{2}\right\rangle|0\rangle + \cos(\theta)\left|\theta + \frac{\pi}{2}\right\rangle|1\rangle = \left|\theta + \frac{\pi}{2}\right\rangle\left|\theta + \frac{\pi}{2}\right\rangle.$$

(d) Assume that system A has obtained the state $|\theta\rangle$ after the measurement from part (c).

Let $\mathcal{H}_B$ have an orthonormal basis $\{|\phi\rangle, \left|\phi + \frac{\pi}{2}\right\rangle\}$ for some $\phi \in [0, \pi)$, where the basis elements are defined similar to (1) and (2). Show that when B performs a von Neumann measurement on its qubit from the state $|\theta\rangle$, then the outcome is "ϕ" with probability $\cos^2(\theta - \phi)$ and the outcome is "$\phi + \frac{\pi}{2}$" with probability $\sin^2(\theta - \phi)$. Here it can be useful to note that $R_y(2\phi)^\dagger = R_y(-2\phi)$ and to recall that any unitary operator preserves the inner product.

Problem 2. Let H be the Hadamard gate, and X and Z be the Pauli gates

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}, X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

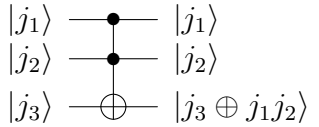
(a) Let CU be the controlled-U 2-qubit gate, for U a unitary 1-qubit gate. When $U = X$, we use the standard notation $CNOT$ for CX. Show that for $U = Z$, the two circuits below for CZ are equivalent:



(b) For a unitary 1-qubit gate U define the CCU gate (controlled-controlled-U) gate by its action on a 3-qubit basis element $|j_1 j_2 j_3\rangle$ for $j_1, j_2, j_3 \in \{0, 1\}$ given as follows

$$CCU\,|j_1 j_2 j_3\rangle = \begin{cases} |j_1 j_2\rangle\,(U\,|j_3\rangle), & \text{if } j_1 = j_2 = 1 \\ |j_1 j_2 j_3\rangle, & \text{in all other cases.} \end{cases}$$

Verify that the $CCNOT$ gate corresponding to $U = X$ is implemented by the following circuit:



Write a circuit for CCZ that uses $CCNOT$ and two gates H, similar to the circuit from part (a) that implements CZ. Verify that your circuit implements CCZ by applying it to input states $|j_1 j_2 j_3\rangle$ from the standard basis of the 3-qubit system.