UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in	MAT 3420 — Quantum computing
Day of examination:	Thursday, June 15, 2023
Examination hours:	9.00 – 13.00
This problem set consists of 4 pages.	
Appendices:	None.
Permitted aids:	None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Note: For each problem, the number in parentheses is the maximum number of points that can be achieved. You are expected to provide a justification for all your answers.

Problem 1

Suppose we have a qubit system with basis $\{|0\rangle, |1\rangle\}$. By H we denote the Hadamard gate given by

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix},$$

and we let $|+\rangle = H|0\rangle$, $|-\rangle = H|1\rangle$.

Suppose the qubit is in the state

$$|\psi\rangle = \frac{-i}{2}|0\rangle + \frac{\sqrt{3}}{2}|1\rangle.$$

1a

(10 points.) Find the probabilities of getting $|0\rangle$ and $|1\rangle$ by measurement of the qubit in the basis $\{|0\rangle, |1\rangle\}$.

1b

(10 points.) Find the probabilities of getting $|+\rangle$ and $|-\rangle$ by measurement of the qubit in the basis $\{|+\rangle, |-\rangle\}$.

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Problem 2

Consider a 2-qubit system with its computational basis elements $|00\rangle, |01\rangle, |10\rangle, |11\rangle$.

2a

(10 points.) Suppose the system is in the state

$$|\psi\rangle = \frac{1}{2\sqrt{2}}|00\rangle - \frac{1}{2\sqrt{2}}|01\rangle - \frac{\sqrt{3}}{2\sqrt{2}}|10\rangle + \frac{\sqrt{3}}{2\sqrt{2}}|11\rangle .$$

Write down the Partial Measurement Rule for this state of the system when we measure each qubit separately, indicating the probabilities of getting $|0\rangle$ or $|1\rangle$. For each possible outcome of the single qubit, write down the outcome state of the combined system. Is $|\psi\rangle$ a product state or an entangled state?

2b

(10 points.) Suppose the system is in the state

$$|\varphi\rangle = \frac{1}{\sqrt{3}}\big(|00\rangle - i\,|10\rangle + |11\rangle\big).$$

Determine if this is an entangled state.

Problem 3

3a

(5 points) Give the definition of the density matrix of a mixed state of a quantum system.

3b

(5 points) Consider a qubit system. Find the density matrix of a mixed state that is an equal mixture of $|u\rangle$ and $|v\rangle$ defined as

$$|u\rangle = \frac{2}{\sqrt{5}}|0\rangle - \frac{1}{\sqrt{5}}|1\rangle$$

and

$$|v\rangle = \frac{2}{\sqrt{5}}|0\rangle + \frac{1}{\sqrt{5}}|1\rangle .$$

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Problem 4

Let A and B be two laboratories which share the GHZ state (a 3-qubit state)

$$|\varphi\rangle = \frac{|000\rangle + |111\rangle}{\sqrt{2}},$$

where the first two qubits, listed from left to right, are with A, and the last one is with B. Suppose A wants to send a state $|\psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$ to B through quantum teleportation, thus the combined state of the system is

$$|\psi\rangle \otimes |\varphi\rangle = \frac{\alpha\,|0000\rangle + \alpha\,|0111\rangle + \beta\,|1000\rangle + \beta\,|1111\rangle}{\sqrt{2}},$$

with the qubits numbered as $1, 2, 3, 4$ from left to right.

4a

(10 points) We denote $\mathrm{CNOT}_{i\mapsto j}$ the controlled-NOT gate with qubit i as control and j as target, for $1 \leq i, j \leq 4$. Let A apply $\mathrm{CNOT}_{2\mapsto 3}$ to its qubits, followed by $\mathrm{CNOT}_{1\mapsto 2}$ and by H on A's first qubit. Finally let A measure its three qubits. Write a quantum circuit for this sequence of transformations on the system.

4b

(10 points) Determine the state $|\eta\rangle$ of the combined system after applying the gates of the quantum circuit in part 4a, and before A's measurement. Let A measure its three qubits in the basis $\{|0\rangle, |1\rangle\}$. Find the probability that A sees $|000\rangle$ and, assuming that A sees $|000\rangle$, write down the resulting state of the combined system.

4c

(10 points) Find all the possible basis states of $(\mathbb{C}^2)^{\otimes 3}$ that A can see by measurement of its qubits from the state $|\eta\rangle$ with the corresponding probabilities. In each case, assuming that A communicates the result to B via a classical channel, determine the unitary gates that B must apply in order to recover the state $|\psi\rangle$.

Problem 5

Consider Grover's quantum search algorithm. Suppose that we have a function $f : \{0, 1, \ldots, N-1\} \to \{0, 1\}$, where $N = 2^n$ for a positive integer $n \geq 1$, and assume that there is a unique marked item x^* with $0 \leq x^* \leq N-1$ and $f(x^*) = 1$. Write integers $x \in \{0, 1, \ldots, N-1\}$ in binary representation corresponding to n-bit strings.

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5a

(10 points) Let $|0\rangle, |1\rangle, \ldots, |N-1\rangle$ be the basis states in $\mathbb{C}^N$ (where $\mathbb{C}^N$ can be identified with $(\mathbb{C}^2)^{\otimes n}$), consider the uniform superposition state

$$|\phi\rangle = H^{\otimes n}|0\rangle = \frac{1}{\sqrt{N}} \sum_{x=0}^{N-1} |x\rangle,$$

and let U_f be the unitary phase query with $U_f|y\rangle = (-1)^{f(y)}|y\rangle$ for $y \in \{0, 1, \ldots, N-1\}$. Consider further the diffusion operator D on $\mathbb{C}^N$, where

$$D|x\rangle = \frac{2}{N} \sum_{z=0}^{N-1} |z\rangle - |x\rangle, \text{ for } z \in \{0, 1, \ldots, N-1\}.$$

Show that $|\phi\rangle$ belongs to a two-dimensional real vector space spanned by $|x^*\rangle$ and a unit vector $|\overline{x}\rangle$ that is an equal superposition over the unmarked items. Describe the states $DU_f|\phi\rangle$ and $(DU_f)^2|\phi\rangle$ in terms of reflections in this space. Determine the probability of getting $|x^*\rangle$ by measurement of each of these states in the basis $\{|x^*\rangle, |\overline{x}\rangle\}$.

5b

(10 points) Argue that if we measure in the basis $\{|x^*\rangle, |\overline{x}\rangle\}$ after t iterations of applying DU_f, the probability of getting $|x^*\rangle$ is close to 1 for t of order $\sqrt{N}$ (a constant multiple of $\sqrt{N}$).

END.