MAT3420 Exam 2024 - Suggested solution

Problem 1 For a qubit system in the state

$$|\psi\rangle = \frac{2+3i}{5}|0\rangle - \frac{2\sqrt{3}i}{5}|1\rangle,$$

a measurement in the computational basis outputs "0" with probability $|\frac{2+3i}{5}|^2 = \frac{13}{25}$ and "1" with probability $|\frac{2\sqrt{3}i}{5}|^2 = \frac{12}{25}$. Since iZ is a unitary operator, as may be verified from $(iZ)^*(iZ) = I$, it preserves the inner product. Therefore $|u\rangle$ and $|v\rangle$ form an orthonormal basis. By a measurement in this basis we get output "u" with probability$|\langle u|\psi\rangle|^2 = |i|^2|\frac{2+3i}{5}|^2 = \frac{13}{25}$ and output "v" with probability $|-i\frac{2\sqrt{3}i}{5}|^2 = \frac{12}{25}$.

Problem 2

2a. Suppose a bipartite system is in the state

$$|\psi\rangle = \frac{1-i}{2}|01\rangle + \frac{1+i}{2}|10\rangle.$$

If a measurement is performed on the system in the basis $\{|0\rangle, |1\rangle\}$ of $\mathcal{H}_A$, the outcome is "0" with probability $|\frac{1-i}{2}|^2 = \frac{1}{2}$ and the output state is $|01\rangle$, and we get outcome "1" with probability $|\frac{1+i}{2}|^2 = \frac{1}{2}$, leaving the system in the state $|01\rangle$.

2b. The density operator ρ^{AB} of the state

$$|\varphi\rangle = \frac{1}{\sqrt{2}}\big(i|01\rangle + |11\rangle\big)$$

is defined as $\rho^{AB} = |\psi\rangle\langle\psi|$ and it is equal to the matrix

$$\begin{bmatrix} 0 & 0 & 0 & 0 \\ 0 & 1/2 & 0 & i/2 \\ 0 & 0 & 0 & 0 \\ 0 & -i/2 & 0 & 1/2 \end{bmatrix}.$$

Problem 3

3a. The phase estimation problem on input state $|\psi\rangle$ outputs the phase ω. In case that $\omega = 0.x_1x_2\ldots x_n$, the quantum algorithm means application of the unitary transformation

$$QFT_{2^n}^{-1} : |\psi\rangle = \frac{1}{\sqrt{2^n}}\sum_{y=0}^{2^n-1} e^{2\pi i\frac{x}{2^n}y}|y\rangle \mapsto |x_n\ldots x_2x_1\rangle,$$

where x is the integer so that $\omega = \frac{x}{2^n}$. In this case the algorithm recovers exactly the integer x from the output $|x_n\ldots x_2x_1\rangle$, either by logically relabelling the qubits or by performing SWAP gates to bring the output state in the form $|x_1x_2\ldots x_n\rangle$. For an arbitrary $\omega \in (0,1)$, the algorithm outputs

an estimate $\tilde{\omega} = \frac{x}{2^n}$, which with high probability is closest to ω up to a prescribed small error.

3b. This is a direct verification giving values $x = 0$ and $x = 1$ in $H|\psi\rangle$, where $|\psi\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + (-1)^x |1\rangle\big)$.

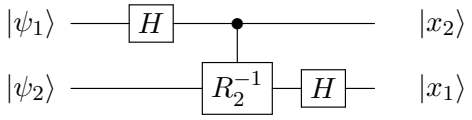
3c. The inverse is determined by

$$R_2^{-1} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-2\pi i/2^2} \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & e^{-2\pi i(0.01)} \end{bmatrix}.$$

When $n = 2$ we have, as shown in Lecture 16 (see also section 7.1 in the lecture book by Kaye, Laflamme, Mosca) that

$$|\psi\rangle = \frac{1}{\sqrt{2}}\big(|0\rangle + e^{2\pi i(0.x_2)} |1\rangle\big) \otimes \frac{1}{\sqrt{2}}\big(|0\rangle + e^{2\pi i(0.x_2x_1)} |1\rangle\big).$$

Then $(H \otimes I)|\psi\rangle = |x_2\rangle \otimes \frac{1}{\sqrt{2}}\big(|0\rangle + e^{2\pi i(0.x_1x_2)} |1\rangle\big)$, where we used the result of problem 3b in the first qubit. Next we apply the gate c-(R_2^{-1}), so when $x_2 = 1$ the effect is to apply R_2^{-1} on $e^{2\pi i(0.x_1 1)} |1\rangle$, giving the result $e^{2\pi i(0.x_1 1 - 0.01)} |1\rangle = e^{2\pi i(0.x_1)}$. Finally, we apply identity in the first qubit and an H gate in the second qubit to output the state $|x_2\rangle |x_1\rangle$, as wanted for the phase estimation algorithm when $n = 2$. The circuit is described below:



3d. Now we assume $n = 3$ and $\omega = 0.x_1x_2x_3$. Then similarly to problem 3c we write $|\psi\rangle$ in product form as

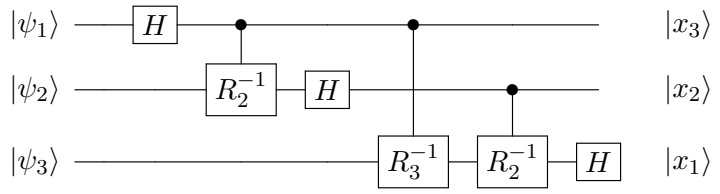
$$|\psi\rangle = \otimes_{k=1}^{3} \left(\frac{|0\rangle + e^{2\pi i(2^{3-k}\omega)} |1\rangle}{\sqrt{2}} \right)$$

or, further, as the product $|\psi_1\rangle |\psi_2\rangle |\psi_3\rangle$ given by

$$\left(\frac{|0\rangle + e^{2\pi i(0.x_3)} |1\rangle}{\sqrt{2}} \right) \otimes \left(\frac{|0\rangle + e^{2\pi i(0.x_2x_3)} |1\rangle}{\sqrt{2}} \right) \otimes \left(\frac{|0\rangle + e^{2\pi i(0.x_1x_2x_3)} |1\rangle}{\sqrt{2}} \right).$$

Then we use a circuit where the first round of gates applies $H \otimes I \otimes I$, giving output $|x_3\rangle |\psi_2\rangle |\psi_3\rangle$, followed by c-$(R_2^{-1}) \otimes I$ with the first qubit controlling the second qubit, followed by an H gate in the second (middle) qubit to output $|x_3\rangle |x_2\rangle |\psi_3\rangle$. Finally, a controlled gate c-R_3^{-1} with first qubit as control and third qubit as target followed by c-$(R_2^{-1}) \otimes I$ with the second qubit controlling the third and an H gate in the third qubit to output

$|x_3\rangle\,|x_2\rangle\,|x_1\rangle$. Thus, the circuit is



3e. The Quantum Fourier transform is the inverse of the phase estimation transformation $\mathrm{QFT}_{2^n}^{-1}$. Thus, it uses the same number of gates. For $n = 2$ we use $1 + 2$ gates, when $n = 3$ we use $1 + 2 + 3$ gates. In general, the algorithm uses $1 + 2 + \cdots + n = n(n + 1)/2$ gates, which means $\mathcal{O}(n^2)$.

Problem 4

4a. Verify directly that $H\,|x\rangle = \frac{1}{\sqrt{2}}(|0\rangle + (-1)^x\,|1\rangle)$. When $n = 2$ compute the tensor product state

$$\frac{1}{\sqrt{2}}\big(|0\rangle + (-1)^{x_1}\,|1\rangle\big) \otimes \frac{1}{\sqrt{2}}\big(|0\rangle + (-1)^{x_2}\,|1\rangle\big)$$

by expanding the parentheses to obtain $\frac{1}{\sqrt{2^2}} \sum_{\mathbf{z}\in\{0,1\}^2} (-1)^{z_1x_1+z_2x_2}\,|z_1\rangle\,|z_2\rangle$.

4b. Applying the sequence of gates as indicated in the problem text, see Lecture 13, the input state $|\psi\rangle$ is transformed to

$$|\eta\rangle = \frac{1}{2^n} \sum_{\mathbf{z}\in\{0,1\}^n} \left(\sum_{\mathbf{x}\in\{0,1\}^n} (-1)^{f(\mathbf{x})+\mathbf{x}\cdot\mathbf{z}} \right) |\mathbf{z}\rangle \, \big(\frac{|0\rangle - |1\rangle}{\sqrt{2}}\big).$$

Examine the possibilities for outcome of a measurement in the input register. The amplitude of $|\mathbf{z}\rangle = |0\rangle^{\otimes n}$ is equal to

$$\frac{1}{2^n} \sum_{\mathbf{x}\in\{0,1\}^n} (-1)^{f(\mathbf{x})}.$$

There are two possibilities for outcome of a measurement: we get outcome "$00\ldots0$", corresponding to the state $|0\rangle$, and this can only have happened if f is constant, meaning $f(\mathbf{x}) = 0$ for all strings $\mathbf{x}$ or $f(\mathbf{x}) = 1$ for all strings $\mathbf{x}$, because in this case the amplitude above is non-zero. Or we get outcome "$z_1z_2\ldots z_n$" where not all of the z_j's are 0, and this can only happen when f is balanced. That is because $\sum_{\mathbf{x}\in\{0,1\}^n}(-1)^{f(\mathbf{x})} = 2^{n-1}\cdot 1 + 2^{n-1}\cdot(-1) = 0$, making the amplitude of $|0\rangle$ equal to 0. Thus, one quantum query followed by a measurement in the first register determines the form of f.