UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in: MAT 3420 — Quantum computing

Day of examination: Thursday 05 June, 2025

Examination hours: 15.00 – 19.00

This problem set consists of 7 pages.

Appendices: None

Permitted aids: None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Determine for each of the following statements whether it is true or not. Give a short argument for your answer.

1a

(5 pts.) There is a unitary operator $U : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$ acting as $U(|x\rangle \otimes |0\rangle) = |x\rangle \otimes |x\rangle$ for $x \in \{0, 1\}$.

Solution: True. The CNOT gate does the job.

1b

(5 pts.) The set $\{H, T, CZ\}$ consisting of the Hadamard gate, the T-gate and the controlled-Z gate is a universal set of quantum gates.

Solution: True. We have $\text{CNOT} = (\mathbb{1}_2 \otimes H)CZ(\mathbb{1}_2 \otimes H)$ and we know from the lecture that $\{H, T, \text{CNOT}\}$ is universal.

1c

(5 pts.) There is a unitary operator $U : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$ acting as $U(|\psi\rangle) = |\phi\rangle$ for $|\psi\rangle = (|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)/\sqrt{2}$ and $|\phi\rangle = |0\rangle \otimes |1\rangle$.

Solution: True. Either we can extend $|\psi\rangle$ and $|\phi\rangle$ to ONBs and build the unitary operator explicitly, or we note that the unitary operator

$$U = (H \otimes X)\text{CNOT},$$

does the job.

(Continued on page 2.)

1d

(5 pts.) The quantum state $|\alpha\rangle \in \mathbb{C}^3 \otimes \mathbb{C}^3$ given by

$$|\alpha\rangle = \frac{1}{\sqrt{5}}(|0\rangle \otimes |0\rangle + |0\rangle \otimes |1\rangle + |1\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle + |2\rangle \otimes |2\rangle),$$

is entangled.

Solution: True. We use the method from the lecture. The quantum state corresponds to the matrix

$$\frac{1}{\sqrt{5}} \begin{pmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix},$$

which has rank 2 (span of the columns/rows is clearly 2-dimensional). Thereby, the Schmidt-rank of $|\psi\rangle$ is 2 and $|\psi\rangle$ is entangled.

1e

(5 pts.) There are unitary operators $U_1, U_2 : \mathbb{C}^2 \to \mathbb{C}^2$ such that $(U_1 \otimes U_2)(|\psi\rangle) = |\phi\rangle$ for $|\psi\rangle = (|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)/\sqrt{2}$ and $|\phi\rangle = |0\rangle \otimes |1\rangle$.

Solution: False. For any pair of unitary operators $U_1, U_2 : \mathbb{C}^2 \to \mathbb{C}^2$ the state $(U_1 \otimes U_2)|\psi\rangle$ has Schmidt-rank 2, and hence it cannot equal the product state $|\phi\rangle$.

1f

(5 pts.) Consider two qubits in the quantum state $|\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ given by

$$|\psi\rangle = (H \otimes H)\mathrm{CNOT}(H \otimes \mathbb{1}_2)(|0\rangle \otimes |0\rangle).$$

If $m_1, m_2 \in \{0, 1\}$ are the measurement outcomes when measuring each of the qubits in the computational basis $\{|0\rangle, |1\rangle\}$, then we always have $m_1 = m_2$.

Solution: True. The circuit first prepares the maximally entangled that $|\psi\rangle = (|0\rangle \otimes |0\rangle + |1\rangle \otimes |1\rangle)/\sqrt{2}$ and then applies a Hadamard gates to both qubits. By an identity from the lecture we have $(H \otimes H)|\psi\rangle = |\psi\rangle$ and hence the measurements are just applied to each half of a maximally entangled state. Clearly, the measurement outcomes are perfectly correlated then.

Problem 2

(10 pts.) Consider the quantum state $|\phi_{\text{magic}}\rangle \in \mathbb{C}^2$ given by

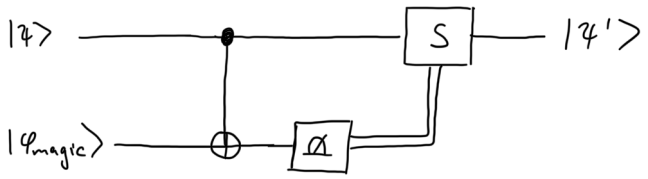
$$|\varphi_{\text{magic}}\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{i\pi/4}|1\rangle\right),$$

(Continued on page 3.)

and recall the single-qubit gates

$$S = \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \quad \text{and} \quad T = \begin{pmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{pmatrix}.$$

Consider the following circuit: Here, the state $|\psi\rangle \in \mathbb{C}^2$ is some unknown



input state and the measurement is with respect to the computational basis. Show that $|\psi'\rangle = T|\psi\rangle$ (up to a phase factor), i.e., the circuit implements the T-gate using the magic state as a resource.

Solution: We set $|\psi\rangle = a|0\rangle + b|1\rangle$ for $a, b \in \mathbb{C}$ with $|a|^2 + |b|^2 = 1$. Then, we compute

$$\begin{aligned} &\mathrm{CNOT}(|\psi\rangle \otimes |\varphi_{\text{magic}}\rangle) \\ &\quad = (\frac{a}{\sqrt{2}}|0\rangle + \frac{b}{\sqrt{2}}e^{i\pi/4}|1\rangle) \otimes |0\rangle + (\frac{a}{\sqrt{2}}e^{i\pi/4}|0\rangle + \frac{b}{\sqrt{2}}|1\rangle) \otimes |1\rangle. \end{aligned}$$

If the measurement gives outcome 0, we find the post-measurement state

$$|\psi_0\rangle = \sqrt{2}(\frac{a}{\sqrt{2}}|0\rangle + \frac{b}{\sqrt{2}}e^{i\pi/4}|1\rangle) = a|0\rangle + be^{i\pi/4}|1\rangle = T|\psi\rangle.$$

Since S is not applied in this case, we find that the $|\psi'\rangle = T|\psi\rangle$. If the measurement outcome is 1, then we find the post-measurement state

$$|\psi_1\rangle = \sqrt{2}(\frac{a}{\sqrt{2}}e^{i\pi/4}|0\rangle + \frac{b}{\sqrt{2}}|1\rangle) = ae^{i\pi/4}|0\rangle + b|1\rangle = e^{i\pi/4}S^\dagger T|\psi\rangle.$$

Since S is applied in this case, we find that $|\psi'\rangle = e^{i\pi/4}T|\psi\rangle$ which represents the same quantum state as $T|\psi\rangle$.

Problem 3

Let $x = x_0x_1$ denote a fixed 2-bit string.

3a

(10 pts.) Consider the single-qubit unitary operator $U_x : \mathbb{C}^2 \to \mathbb{C}^2$ acting as

$$U_x|j\rangle = (-1)^{x_j}|j\rangle,$$

(Continued on page 4.)

for $j \in \{0,1\}$. Compute the quantum state $HU_xH|0\rangle$ and compute the probabilities (depending on x) of obtaining the outcomes 0 and 1, when measuring it in the computational basis $\{|0\rangle, |1\rangle\}$.

Solution: We compute

$$HU_xH|0\rangle = H\frac{U_x|0\rangle + U_x|1\rangle}{\sqrt{2}} = H\frac{(-1)^{x_0}|0\rangle + (-1)^{x_1}|1\rangle}{\sqrt{2}}$$
$$= \begin{cases} \pm|0\rangle \text{ if } x_0 = x_1 \\ \pm|1\rangle \text{ if } x_0 \neq x_1 \end{cases}$$

If $x_0 = x_1$, then measuring in the computational basis gives outcome 0 with certainty. If $x_0 \neq x_1$, then measuring in the computational basis gives outcome 1 with certainty.

3b

(10 pts.) Consider now a 2-qubit unitary operator $U'_x : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$ satisfying

$$U'_x(|j\rangle \otimes |0\rangle) = (-1)^{x_j}|j\rangle \otimes |j\rangle.$$

Compute the quantum state $(H \otimes \mathbb{1}_2)U'_x(H \otimes \mathbb{1}_2)(|0\rangle \otimes |0\rangle)$ and compute the probabilities (depending on x) of obtaining the outcomes 0 and 1, when measuring the first qubit in the computational basis.

Solution: We compute

$$U'_x(H\otimes\mathbb{1}_2)(|0\rangle\otimes|0\rangle) = U'_x\frac{|0\rangle \otimes |0\rangle + |1\rangle \otimes |0\rangle}{\sqrt{2}} = \frac{1}{\sqrt{2}}\left((-1)^{x_0}|0\rangle \otimes |0\rangle + (-1)^{x_1}|1\rangle \otimes |1\rangle\right).$$

Then, we obtain

$$(H \otimes \mathbb{1}_2)U'_x(H \otimes \mathbb{1}_2)(|0\rangle \otimes |0\rangle)$$
$$= \frac{1}{\sqrt{2}}\left(|0\rangle \otimes \frac{(-1)^{x_0}|0\rangle + (-1)^{x_1}|1\rangle}{\sqrt{2}} + |1\rangle \otimes \frac{(-1)^{x_0}|0\rangle - (-1)^{x_1}|1\rangle}{\sqrt{2}}\right)$$

Measuring the first qubit in the computational basis gives outcome 0 with probability 1/2 and outcome 1 with probability 1/2 regardless of the values of x_0 and x_1.

Problem 4

Recall the classical Toffoli gate $\text{Tof} : \{0,1\}^3 \to \{0,1\}^3$ acting as $\text{Tof}(b_1, b_2, b_3) = (b_1, b_2, b_3 \oplus \text{AND}(b_1, b_2))$.

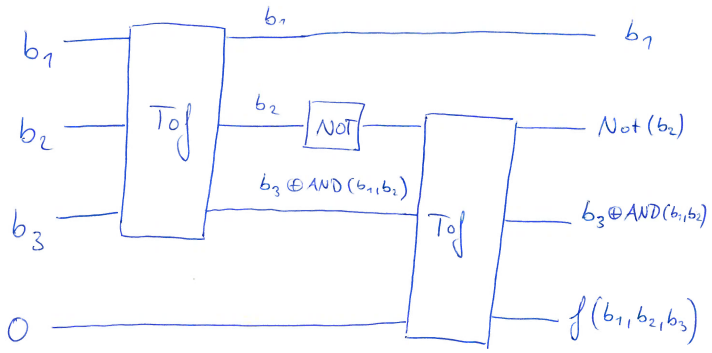
(Continued on page 5.)

4a

(10 pts.) Write down or draw a classical circuit involving two Toffoli gates, a NOT gate and an ancilla bit to compute the Boolean function $f : \{0,1\}^3 \to \{0,1\}$ given by

$$f(b_1, b_2, b_3) = \mathrm{AND}(\mathrm{NOT}(b_2), b_3 \oplus \mathrm{AND}(b_1, b_2)).$$

Solution: We have the following circuit computing the desired function using one ancilla bit:



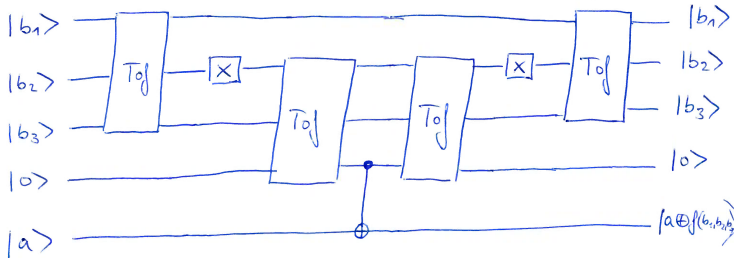
4b

(10 pts.) Let f denote the Boolean function introduced in the previous subexercise. Write down or draw a quantum circuit to implement the unitary operator $U_f : (\mathbb{C}^2)^{\otimes 4} \to (\mathbb{C}^2)^{\otimes 4}$ acting as

$$U_f\big(|b_1\rangle \otimes |b_2\rangle \otimes |b_3\rangle \otimes |a\rangle\big) = |b_1\rangle \otimes |b_2\rangle \otimes |b_3\rangle \otimes |a \oplus f(b_1, b_2, b_3)\rangle,$$

for $a, b_1, b_2, b_3 \in \{0,1\}$. You are not allowed to use measurements. You may assume the quantum version of the Toffoli gate as a black box, and you are allowed to use ancillas initialized in $|0\rangle$. All ancillas have to be returned to their initial state after the execution of the circuit.

Solution: To find the quantum circuit we use the uncomputation trick from the lecture. For this particular circuit we have the quantum circuit:



(Continued on page 6.)

Problem 5

Let $s \in \{0,1\}^n$ denote some unknown bit-string and consider the function $f : \{0,1\}^n \to \{0,1\}$ given by

$$f(x) = x \cdot s = x_1 s_1 + x_2 s_2 + \cdots + x_n s_n \bmod 2,$$

for any $x \in \{0,1\}^n$. Our goal is to find s using a quantum algorithm. Let $U_f : (\mathbb{C}^2)^{\otimes n+1} \to (\mathbb{C}^2)^{\otimes n+1}$ denote the unitary operator acting as

$$U_f(|x\rangle \otimes |y\rangle) = |x\rangle \otimes |y \oplus f(x)\rangle,$$

for any $x \in \{0,1\}^n$ and any $y \in \{0,1\}$.

5a

(10 pts.) Show that $U_f(|x\rangle \otimes |-\rangle) = (-1)^{f(x)}|x\rangle \otimes |-\rangle$ for any $x \in \{0,1\}^n$, where $|-\rangle = (|0\rangle - |1\rangle)/\sqrt{2}$.

Solution: We compute

$$U_f(|x\rangle \otimes |-\rangle) = \frac{U_f(|x\rangle \otimes |0\rangle) - U_f(|x\rangle \otimes |1\rangle)}{\sqrt{2}} = \frac{|x\rangle \otimes |0 \oplus f(x)\rangle - |x\rangle \otimes |1 \oplus f(x)\rangle}{\sqrt{2}}.$$

Considering the two cases we find

$$U_f(|x\rangle \otimes |-\rangle) = \begin{cases} |0\rangle \otimes |-\rangle \text{ if } f(x) = 0 \\ -|0\rangle \otimes |-\rangle \text{ if } f(x) = 1 \end{cases}.$$

This shows that $U_f(|x\rangle \otimes |-\rangle) = (-1)^{f(x)}|x\rangle \otimes |-\rangle$.

5b

(10 pts.) Show that the following quantum circuit computes the bit-string s with probability 1.

Solution: The first line of Hadamards gives

$$|\psi_1\rangle = H^{\otimes n}|0\rangle^{\otimes n} \otimes H|1\rangle = \frac{1}{2^{n/2}} \sum_{x \in \{0,1\}^n} |x\rangle \otimes |-\rangle.$$

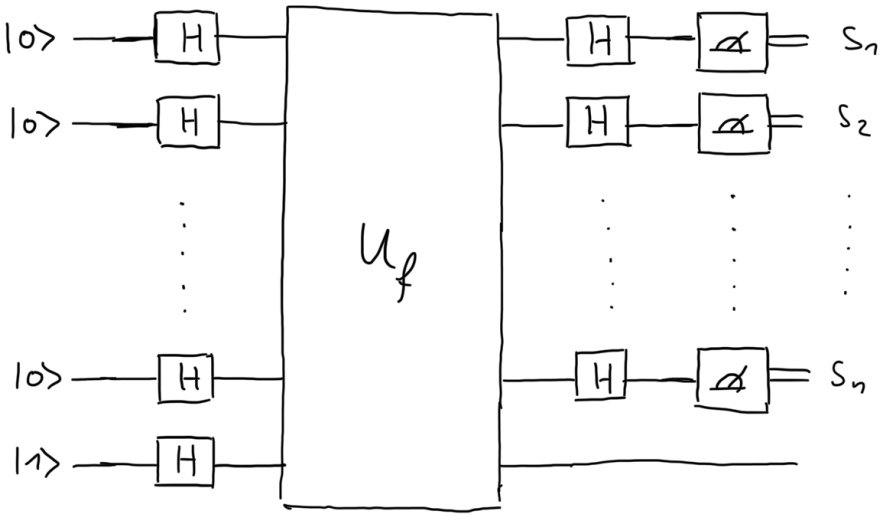
Applying U_f gives the state

$$|\psi_2\rangle = U_f|\psi_1\rangle = \frac{1}{2^{n/2}} \sum_{x \in \{0,1\}^n} U_f(|x\rangle \otimes |-\rangle) = \frac{1}{2^{n/2}} \sum_{x \in \{0,1\}^n} (-1)^{x \cdot s}|x\rangle \otimes |-\rangle,$$

where we used the first exercise and that $f(x) = x \cdot s$ for the unknown string s. Now, either remember from the lecture or compute that

$$H^{\otimes n}|s\rangle = \frac{1}{2^{n/2}} \sum_{x \in \{0,1\}^n} (-1)^{x \cdot s}|x\rangle.$$

(Continued on page 7.)



With this we find

$$|\psi_3\rangle = (H^{\otimes n} \otimes \mathbb{1}_2)|\psi_2\rangle = |s\rangle \otimes |-\rangle.$$

Measuring the first n qubits in the computational basis gives the outcome s with certainty. (This quantum circuit is identical to the Deutsch-Josza algorithm. In this particular context it is called the Bernstein-Vazirani algorithm.)