UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in:	MAT 3420 — Quantum computing (Mock exam)
Day of examination:	Whenever
Examination hours:	xx.xx−xx+4.xx
This problem set consists of 5 pages.	
Appendices:	Matlab calculation on last page.
Permitted aids:	None.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Determine for each of the following statements whether it is true or not. Give a short argument for your answer.

1a

(5 pts.) The vectors $\frac{1}{\sqrt{5}}\begin{pmatrix}1\\2i\end{pmatrix}$ and $\frac{1}{\sqrt{10}}\begin{pmatrix}1+i\\-2+2i\end{pmatrix}$ represent the same quantum state.

1b

(5 pts.) Alice and Bob each have a 2-dimensional quantum system in their possession and each of them can perform a 2-outcome measurement on the system in their posession. If Bob's measurement outcome is always the same as Alice's measurement outcome, then this means that the joint state of the two systems is entangled.

1c

(5 pts.) Let $\hat{n}, \hat{m} \in \mathbb{R}^3$ be non-parallel unit vectors. Then, any single qubit unitary U may be written as

$$U = e^{i\alpha} R_{\hat{n}}(\beta) R_{\hat{m}}(\gamma) R_{\hat{n}}(\delta),$$

for appropriate choices α, β, γ and δ.

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1d

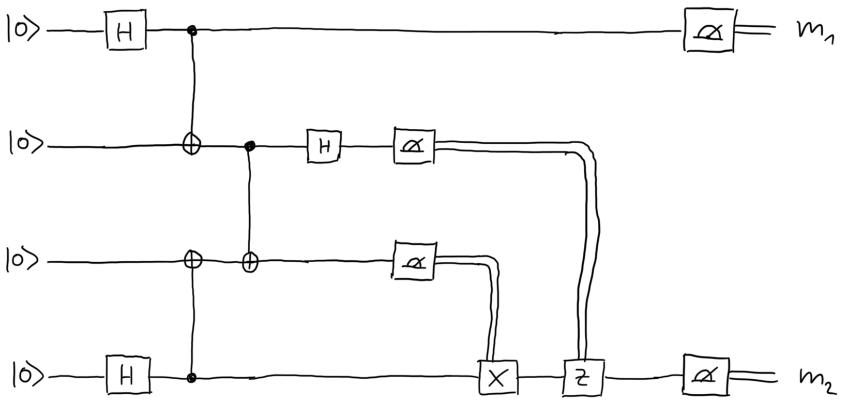
(5 pts.) There exist quantum states $|\psi\rangle, |\phi\rangle \in \mathbb{C}^2$ such that the quantum state $\text{CNOT}(|\psi\rangle \otimes |\phi\rangle)$ is entangled.

1e

(5 pts.) If a decision problem is solvable by a quantum computer, then it is also solvable by a classical computer.

1f

(5 pts.) Consider the following circuit.



Here, all measurements are in the computational basis $\{|0\rangle, |1\rangle\}$. We always have $m_1 = m_2$.

Problem 2

(10 pts.) Consider the swap gate $\text{SWAP} : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$ given by

$$\text{SWAP}(|a\rangle \otimes |b\rangle) = |b\rangle \otimes |a\rangle,$$

for $a, b \in \{0, 1\}$ and extended linearily. Implement the swap gate using only CNOT gates (you are allowed to use either of the two qubits as control, but write an explicit circuit).

Problem 3

(10 pts.) Consider the 3-qubit quantum state $|\psi\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$ given by

$$|\psi\rangle = \frac{1}{4}\left(|000\rangle + 2|110\rangle + |010\rangle + 3|101\rangle + |011\rangle\right),$$

where we use the notation $|ijk\rangle = |i\rangle \otimes |j\rangle \otimes |k\rangle$ for $i, j, k \in \{0, 1\}$. Suppose the third qubit is measured in the computational basis $\{|0\rangle, |1\rangle\}$. Determine

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the probabilities for obtaining outcome 0 and outcome 1. Determine the post-measurement state on the first two qubits conditioned on the measurement outcome being 0. Is this state entangled?

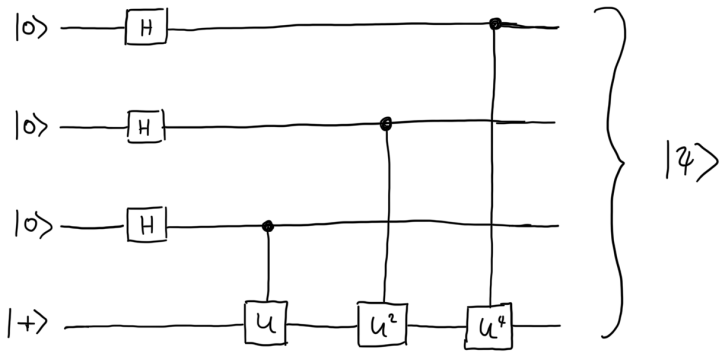
Problem 4

Consider the unitary operator $U : \mathbb{C}^2 \to \mathbb{C}^2$ given by

$$U = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}.$$

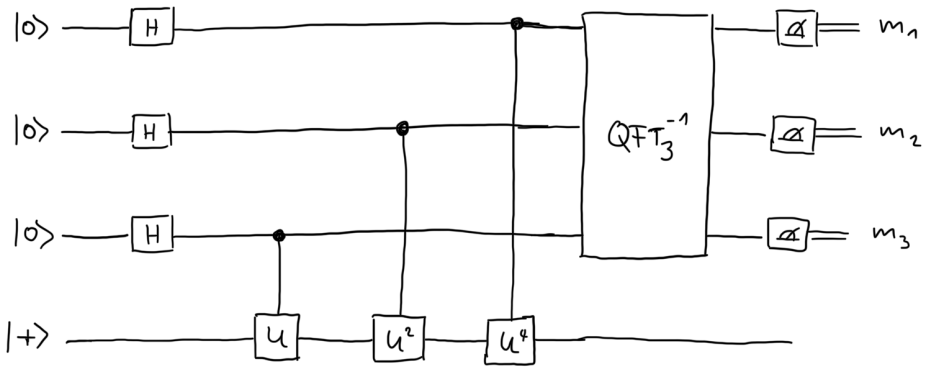
4a

(10 pts.) Consider the following circuit and compute the output state $|\psi\rangle \in (\mathbb{C}^2)^{\otimes 4}$.



4b

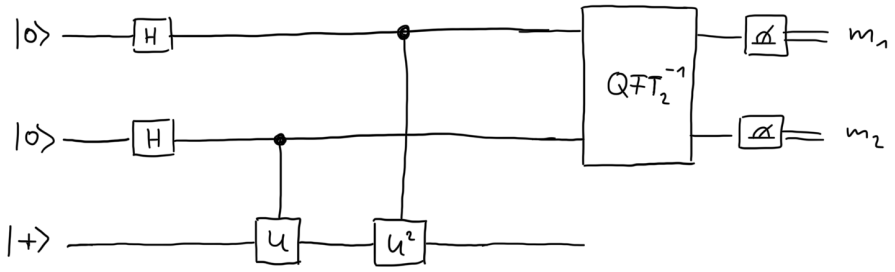
(10 pts.) Compute the quantum state arising from applying the inverse quantum Fourier transform $\mathrm{QFT}_3^{-1} : (\mathbb{C}^2)^{\otimes 3} \to (\mathbb{C}^2)^{\otimes 3}$ to the first three qubits. What is the probability for obtaining the measurement outcome $(m_1, m_2, m_3) = (0, 0, 1)$ in the following circuit?



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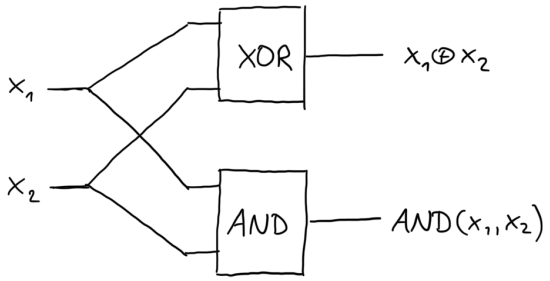
4c

(10 pts.) Consider the following circuit and compute the probabilities for obtaining the outcomes $(m_1, m_2) = (0, 0)$ and $(m_1, m_2) = (0, 1)$.



Problem 5

Consider the following classical circuit implementing the so-called *half-adder*.



Recall the classical Toffoli gate $\mathrm{Tof} : \{0,1\}^3 \to \{0,1\}^3$ acting as $\mathrm{Tof}(b_1, b_2, b_3) = (b_1, b_2, b_3 \oplus \mathrm{AND}(b_1, b_2))$.

5a

(10 pts.) Write down or draw an implementation of the half-adder using only Toffoli gates, Not gates and ancillas.

5b

(10 pts.) Write down or draw a quantum circuit implementing the unitary $U_{HA} : (\mathbb{C}^2)^{\otimes 4} \to (\mathbb{C}^2)^{\otimes 4}$ acting as

$$U_{HA}(|x_1\rangle \otimes |x_2\rangle \otimes |x_3\rangle \otimes |x_4\rangle) = |x_1\rangle \otimes |x_2\rangle \otimes |x_3 \oplus (x_1 \oplus x_2)\rangle \otimes |x_4 \oplus \mathrm{AND}(x_1, x_2)\rangle.$$

You are not allowed to use measurements. You may assume the quantum version of the Toffoli gate as a black box, and you are allowed to use ancillas. All ancillas have to be returned to their initial state after the execution of the circuit.

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Matlab.

```
>> exp(2*pi*i/8)

ans =

   0.7071 + 0.7071i

>> QFT2=[1,1,1,1;1,-i,-1,i;1,-1,1,-1;1,i,-1,-i]/2

QFT2 =

   0.5000 + 0.0000i   0.5000 + 0.0000i   0.5000 + 0.0000i   0.5000 + 0.0000i
   0.5000 + 0.0000i   0.0000 - 0.5000i  -0.5000 + 0.0000i   0.0000 + 0.5000i
   0.5000 + 0.0000i  -0.5000 + 0.0000i   0.5000 + 0.0000i  -0.5000 + 0.0000i
   0.5000 + 0.0000i   0.0000 + 0.5000i  -0.5000 + 0.0000i   0.0000 - 0.5000i

>> QFT2'*[1;exp(2*pi*i/8);exp(2*pi*i/4);exp(2*pi*i*3/8)]/2

ans =

   0.2500 + 0.6036i
   0.2500 + 0.1036i
   0.2500 - 0.1036i
   0.2500 - 0.6036i

>> abs(ans).^2

ans =

    0.4268
    0.0732
    0.0732
    0.4268
```