26th February, 2025

COURSE MAT3420— Quantum computing

Mandatory assignment 1 of 1

Submission deadline

Thursday 13rd March 2025, 14:30 in Canvas (canvas.uio.no).

Instructions

Note that you have **one attempt** to pass the assignment, and any attempt will pass the assignment. If you make an honest attempt, then you will receive feedback on it.

You can choose between scanning handwritten notes or typing the solution directly on a computer (for instance with LaTeX). The assignment must be submitted as a single PDF file.

The submission must contain your name, course and assignment number.

It is expected that you give a clear presentation with all necessary explanations. If your presentation is too messy, then you will not receive any feedback.

Application for postponed delivery

If you need to apply for a postponement of the submission deadline due to illness or other reasons, you have to contact the Student Administration at the Department of Mathematics (e-mail: studieinfo@math.uio.no) no later than the same day as the deadline.

All mandatory assignments in this course must be approved in the same semester, before you are allowed to take the final examination.

Complete guidelines about delivery of mandatory assignments:

uio.no/english/studies/admin/compulsory-activities/mn-math-mandatory.html

GOOD LUCK!

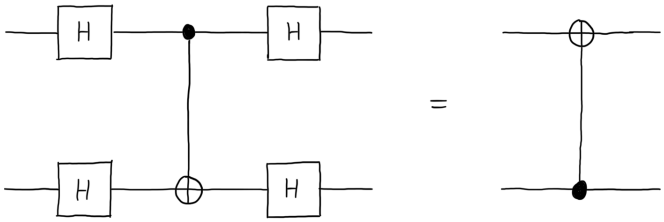
Problem 1. The controlled-NOT gate (CNOT gate) is the unitary $C_{not} : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$ acting as

$$C_{not}(|c\rangle \otimes |t\rangle) = |c\rangle \otimes |t \oplus c\rangle,$$

on computational basis states where $c, t \in \{0, 1\}$ and $\oplus$ denotes binary addition. Here, we call the first qubit the *control* and the second qubit the *target*. Let $H : \mathbb{C}^2 \to \mathbb{C}^2$ denote the Hadamard gate given by

$$H = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

Show that the unitary $(H \otimes H)C_{not}(H \otimes H)$ is another CNOT gate with the roles of control and target reversed as in the following figure.



Problem 2. For $\theta \in [0, 2\pi)$ define the unitary operator

$$U_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Define the states $|\phi_0\rangle = U_\theta|0\rangle$ and $|\phi_1\rangle = U_\theta|1\rangle$.

1. Show that $\sigma_z\sigma_x|\phi_1\rangle = |\phi_0\rangle$ for the Pauli operators

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ and } \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

2. Show that the maximally entangled state can be written as

$$|\Omega_2\rangle = \frac{1}{\sqrt{2}} \left(|\phi_0\rangle \otimes |\phi_0\rangle + |\phi_1\rangle \otimes |\phi_1\rangle\right).$$

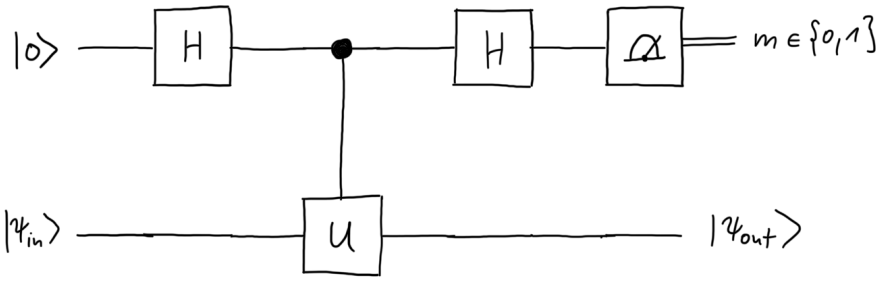
3. Suppose that Alice and Bob share a maximally entangled state. Alice applies the unitary U_θ^{-1} to her share of the state and measures her system in the computational basis. What are the postmeasurement states of Bob's system conditioned on the two measurement outcomes?

4. Suppose that Alice knows the angle $\theta \in [0, 2\pi)$, but Bob does not. Find a protocol using 1 bit of classical communication and a shared maximally entangled state $|\Omega_2\rangle$ so that Bob ends up with the state $|\phi_0\rangle = U_\theta|0\rangle$ in the end.

Problem 3. Consider a selfadjoint unitary operator $U : \mathbb{C}^2 \to \mathbb{C}^2$, i.e., a selfadjoint operator with eigenvalues ± 1. Note that we can think of U as both a unitary gate and as an observable. Define the corresponding controlled unitary $C_U : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$ acting as

$$C_U(|c\rangle \otimes |t\rangle) = |c\rangle \otimes U^c|t\rangle,$$

on product states where $c, t \in \{0, 1\}$, and with $U^c = \mathbb{1}_2$ if $c = 0$ and $U^c = U$ if $c = 1$. Consider the circuit in the following figure:



Here, the controlled unitary C_U is depicted in the usual way (similar to the CNOT from Problem 1) and the measurement is in the computational basis. The state $|\psi_{in}\rangle \in \mathbb{C}^2$ is some unknown input state.

1. Compute the probabilities for obtaining the different measurement outcomes. (Hint: It is helpful to express the state $|\psi\rangle$ in the eigenbasis of U.)

2. Compute the state $|\psi_{out}\rangle$ of the second qubit after obtaining a measurement outcome.

3. Explain in words (brief!) what the circuit is doing.

Problem 4. Let $\vec{v} \in \mathbb{R}^3$ be a real vector with norm $\|\vec{v}\| = 1$ and consider the operator

$$\vec{v} \cdot \vec{\sigma} = v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3,$$

where σ_1, σ_2 and σ_3 are the three Pauli matrices from the lecture. Show that

$$\exp\left(i\theta\vec{v} \cdot \vec{\sigma}\right) = \cos(\theta)\mathbb{1}_2 + i\sin(\theta)\vec{v} \cdot \vec{\sigma}.$$

You may use the following steps.

1. Show that $\vec{v} \cdot \vec{\sigma}$ is a Hermitian and unitary. What are the eigenvalues of $\vec{v} \cdot \vec{\sigma}$?

2. Let $|v_0\rangle$ and $|v_1\rangle$ denote an orthonormal pair of eigenvectors for $\vec{v} \cdot \vec{\sigma}$. How does the spectral decomposition of $\vec{v} \cdot \vec{\sigma}$ look like?

3. Compute $\exp(i\theta \vec{v} \cdot \vec{\sigma})$ thereby verifying the formula from above.