EXERCISES 10

Excercise 1. For vectors $|\psi\rangle, |\phi\rangle \in \mathbb{C}^d$ with $\langle\psi|\psi\rangle = \langle\phi|\phi\rangle = 1$ we introduce the *fidelity* $F(|\psi\rangle, |\phi\rangle) = |\langle\psi|\phi\rangle|$. Show that

$$2(1 - F(|\psi\rangle, |\phi\rangle)) = \min_{\alpha,\beta} \|e^{-i\alpha}|\psi\rangle - e^{-i\beta}|\phi\rangle\|_2^2,$$

where the minimum is over angles $\alpha, \beta \in [0, 2\pi]$, and $\|\cdot\|_2 = \sqrt{\langle\cdot|\cdot\rangle}$ denotes the usual norm on $\mathbb{C}^d$. Use this result to conclude that if two quantum states are close to each other, then their representing vectors have an overlap with modulus close to 1.

Excercise 2 (From previous exam)**.** Consider the following states of a 2-qubit system:

$$\begin{aligned}
|\psi_0\rangle &= \frac{1}{\sqrt{3}}|00\rangle + \sqrt{\frac{2}{3}}|11\rangle, \\
|\psi_1\rangle &= |+-\rangle, \\
|\psi_2\rangle &= \frac{1}{2}\left(|00\rangle + |01\rangle + |10\rangle + |11\rangle\right).
\end{aligned}$$

Determine which of these states are entangled?

Excercise 3 (From previous exam)**.** Let Alice and Bob do an experiment involving the 3-qubit state

$$|\phi\rangle = \frac{1}{\sqrt{2}}\left(|000\rangle + |111\rangle\right),$$

which is also called the *Greenberger-Horne-Zeilinger (GHZ)* state. Let us assume that Alice has two of the three qubits in her laboratory and Bob has the remaining one in his laboratory far away from Alice's. In the following Alice wants to send a state $|\psi\rangle = a|0\rangle + \beta|1\rangle$ of a third qubit in her possession to Bob via a teleportation protocol. Initially, the state of all four qubits is

$$|\psi\rangle \otimes |\phi\rangle = \frac{1}{\sqrt{2}}\left(\alpha|0000\rangle + \alpha|0111\rangle + \beta|1000\rangle + \beta|1111\rangle\right),$$

where the first three qubits are held by Alice and the final one by Bob. We number the qubits $1, 2, 3, 4$ from left to right.

(1) For $1 \leqslant i, j \leqslant 4$, we denote by $CNOT_{i\to j}$ the controlled-NOT gate with qubit i as control and qubit j as target. Let Alice apply $CNOT_{2\to 3}$ to her qubits followed by $CNOT_{1\to 2}$ and then by the gate H on the first qubit. Finally, Alice measures her qubits in the computational basis. Write a quantum circuit for this sequence of transformations.

(2) Determine the state $|\eta\rangle$ of the combined system after applying all the gates of the quantum circuit in part (a), before the measurement. Find the probability that Alice sees gets outcome 000 and assuming she gets this outcome write down the resulting state of the combined system.

(3) Find all possible measurement outcomes Alice can obtain and determine the corresponding probabilities. In each case assume that Alice communicates the outcome to Bob find a unitary gate so that Bob recovers the state $|\psi\rangle$ on his qubit.