EXERCISES 2

Here are a couple of exercises from the chapter on linear algebra in Nielsen and Chuang that are of particular importance.

Excercise 1. Consider the vector $|+\rangle = (|0\rangle + |1\rangle)/\sqrt{2} \in \mathbb{C}^2$. Compute $|+\rangle^{\otimes 2}$ and $|+\rangle^{\otimes 3}$. Write them first in vector form and then expanded in an orthonormal basis of $\mathbb{C}^2 \otimes \mathbb{C}^2$ and $\mathbb{C}^2 \otimes \mathbb{C}^2 \otimes \mathbb{C}^2$, respectively.

Excercise 2.

(1) Show that the tensor product of two Hermitian operators is Hermitian.
(2) Show that the tensor product of two unitary operators is unitary.
(3) Show that the tensor product of two projections is a projection.

Excercise 3. Let $\vec{v} \in \mathbb{R}^3$ be a real vector with norm $\|\vec{v}\| = 1$ and consider the operator

$$\vec{v} \cdot \vec{\sigma} = v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3,$$

where σ_1, σ_2 and σ_3 are the three Pauli matrices from the lecture. Show that

$$\exp(i\theta\vec{v} \cdot \vec{\sigma}) = \cos(\theta)\mathbb{1}_2 + i\sin(\theta)\vec{v} \cdot \vec{\sigma}.$$

You may use the following steps.

(1) Show that $\vec{v} \cdot \vec{\sigma}$ is a Hermitian and unitary. Find the eigenvalues of $\vec{v} \cdot \vec{\sigma}$.
(2) Let $|v_0\rangle$ and $|v_1\rangle$ denote an orthonormal pair of eigenvectors for $\vec{v} \cdot \vec{\sigma}$. How does the spectral decomposition of $\vec{v} \cdot \vec{\sigma}$ look like.
(3) Use the trick from the lecture to compute $\exp(i\theta\vec{v} \cdot \vec{\sigma})$.

Excercise 4. Recall the Hadamard operator

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix},$$

from the lecture. Show that

$$H^{\otimes n} = \frac{1}{\sqrt{2}} \sum_{x,y \in \{0,1\}^n} (-1)^{x \cdot y} |x\rangle\langle y|.$$

The next exercise is to practice tensor products and diagonalization.

Excercise 5. Consider the following tensor products of Pauli operators

$$\sigma_{xx} = \sigma_x \otimes \sigma_x$$
$$\sigma_{yy} = \sigma_y \otimes \sigma_y$$
$$\sigma_{zz} = \sigma_z \otimes \sigma_z.$$

(1) Write down the matrix representations of σ_{xx}, σ_{yy} and σ_{zz} in the computational basis.
(2) Show that $[\sigma_{ii}, \sigma_{jj}] = 0$ for all i, j.
(3) Find a simultaneous diagonalization of the operators σ_{ii} for $i \in \{1, 2, 3\}$.