EXERCISES 3

Excercise 1. For $\theta \in [0, 2\pi)$ define the unitary operator

$$U_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Define the states $|\phi_0\rangle = U_\theta|0\rangle$ and $|\phi_1\rangle = U_\theta|1\rangle$. Show that $\sigma_z\sigma_x|\phi_1\rangle = |\phi_0\rangle$ for the Pauli operators

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ and } \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

Excercise 2. A quantum system evolves according to the Hamiltonian $H = \omega\sigma_x$, where $\omega \in \mathbb{R}$ is an unknown parameter. Initially, the quantum system is in the computational basis state $|0\rangle \in \mathbb{C}^2$.

(1) Find the quantum state $|\psi_t\rangle$ after time $t \in \mathbb{R}$.
(2) After time t, we perform a measurement in the computational basis. What are the probabilities of observing the two outcomes depending on t.
(3) How could an experimental physicist figure out the value of ω?

Excercise 3. Let $\vec{v} \in \mathbb{R}^3$ denote a unit vector and recall (or prove) that the operator

$$\vec{v} \cdot \vec{\sigma} = v_1\sigma_1 + v_2\sigma_2 + v_3\sigma_3,$$

has eigenvalues ± 1. Here, we use the three Pauli operators $\sigma_1 = \sigma_x$, $\sigma_2 = \sigma_y$, and $\sigma_3 = \sigma_z$.

(1) Show that the operators $P_+ = (\mathbb{1}_2 + \vec{v} \cdot \vec{\sigma})/2$ and $P_- = (\mathbb{1}_2 - \vec{v} \cdot \vec{\sigma})/2$ are projections onto the subspaces spanned by eigenvectors of $\vec{v} \cdot \vec{\sigma}$.
(2) Compute the probabilities of obtaining outcomes $+1$ and -1, when measuring the computational basis state $|0\rangle$ using the projective measurement $\{P_+, P_-\}$.

Excercise 4 (The qubit)**.** Consider the state space $\mathbb{C}^2$ of a qubit quantum system.

(1) Show that every unit vector $|\psi\rangle \in \mathbb{C}^2$ can be written as

$$|\psi\rangle = e^{i\gamma}\left(\cos\left(\frac{\theta}{2}\right)|0\rangle + e^{i\phi}\sin\left(\frac{\theta}{2}\right)|1\rangle\right),$$

for angles $\gamma, \theta, \phi \in [0, 2\pi)$. Conclude that every quantum state of the qubit quantum system can be represented by a real vector

$$\vec{v}_\psi = \begin{pmatrix} \cos(\theta) \\ \sin(\theta)\cos(\phi) \\ \sin(\theta)\sin(\phi) \end{pmatrix},$$

on the unit sphere in $\mathbb{R}^3$. This representation of qubit states is called the *Bloch sphere representation.*
(2) Visualize the eigenstates of the Pauli operators σ_x, σ_y and σ_z on the Bloch sphere. How is orthogonality of states represented on the Bloch sphere?
(3) Visualize the action of the unitaries σ_x, σ_y and σ_z on quantum states using the Bloch sphere? Do the same for the Hadamard gate

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$