EXERCISES 4

Excercise 1. Suppose $|\psi\rangle$ is an eigenvector of an Hermitian operator M. What is the expectation value $\mathbb{E}(M)$ and the standard deviation $\Delta(M)$ when measuring the quantum state $|\psi\rangle$ using the observable M.

Excercise 2. Find a projective measurement with two outcomes on $\mathbb{C}^2 \otimes \mathbb{C}^2$ such that measuring the quantum state $|i\rangle \otimes |j\rangle$ gives the sum $i + j$ modulo 2 as an outcome with probability 1. What is the corresponding observable?

Excercise 3. Consider the maximally entangled state $|\Omega_d\rangle \in \mathbb{C}^d \otimes \mathbb{C}^d$ given by

$$|\Omega_d\rangle = \frac{1}{\sqrt{d}} \left(|11\rangle + |22\rangle + \cdots + |dd\rangle \right).$$

(1) Compute the expectation value when measuring the state $|\Omega_2\rangle$ using the observable $\sigma_x \otimes \sigma_z$.
(2) For operators M and N on $\mathbb{C}^d$ show that

$$\langle \Omega_d | M \otimes N | \Omega_d \rangle = \frac{1}{d} \operatorname{Tr}\left[M^T N \right].$$

Hint: Try to prove it first for $d = 2$.
(3) Find a formula for the expectation values when measuring the state $|\Omega_2\rangle$ using the observable $\sigma_i \otimes \sigma_j$ for $i, j \in \{1, 2, 3\}$.

Excercise 4. Consider projections $P_1, \ldots, P_k$ on a finite-dimensional complex Hilbert space V such that

$$P_1 + P_2 + \cdots + P_k = \mathbb{1}_V.$$

Show that $P_i P_j = 0$ when $i \neq j$.