EXERCISES 5

Excercise 1. Suppose three qubits are in the quantum state

$$|\psi\rangle = \frac{1}{2}\left(|000\rangle + |100\rangle + |101\rangle - |111\rangle\right).$$

If we measure the third qubit using the computational basis measurement, what are the probabilities of two outcomes? What is the state of the remaining qubits conditioned on either outcome?

Excercise 2 (The CNOT gate)**.** Consider the CNOT gate given by the matrix

$$U_{cnot} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix},$$

in the standard (lexicographic) ordering of the product basis $\{|ij\rangle\}_{i,j\in\{0,1\}}$.

(1) Show that this operator is unitary and convince yourself that it acts as

$$\begin{aligned} U_{cnot}|00\rangle &= |00\rangle \\ U_{cnot}|01\rangle &= |01\rangle \\ U_{cnot}|10\rangle &= |11\rangle \\ U_{cnot}|11\rangle &= |10\rangle \end{aligned}$$

on the standard basis vectors.

(2) Recall the Hadamard gate

$$H = \frac{1}{\sqrt{2}}\begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Show that the maximally entangled state $|\Omega_2\rangle \in \mathbb{C}^2 \otimes \mathbb{C}^2$ can be written as

$$|\Omega_2\rangle = U_{cnot}(H \otimes \mathbb{1}_{\mathbb{C}^2})|00\rangle.$$

(3) Modify the sequence of gates from the previous exercise to produce the other Bell states from the Bell basis (see lecture).

Excercise 3 (The Deutsch algorithm)**.** Consider a binary function $f : \{0,1\} \to \{0,1\}$ and define a linear operator $U_f : \mathbb{C}^2 \otimes \mathbb{C}^2 \to \mathbb{C}^2 \otimes \mathbb{C}^2$ acting as

$$U_f|ij\rangle = |i(j \oplus f(i))\rangle,$$

for $i, j \in \{0,1\}$, and where $\oplus$ denotes binary addition.

(1) Show that U_f is a unitary.

(2) Consider the operator

$$C_f = (H \otimes \mathbb{1}_{\mathbb{C}^4})U_f(H \otimes H),$$

where H is the Hadamard gate. Compute the state

$$|\psi_f\rangle = C_f\left(|0\rangle \otimes |1\rangle\right).$$

(3) When you measure the first qubit using the computational basis measurement, what do you learn about the function f? Why is this surprising?

Excercise 4. For $\theta \in [0, 2\pi)$ define the unitary operator

$$U_\theta = \begin{pmatrix} \cos(\theta) & -\sin(\theta) \\ \sin(\theta) & \cos(\theta) \end{pmatrix}.$$

Define the states $|\phi_0\rangle = U_\theta|0\rangle$ and $|\phi_1\rangle = U_\theta|1\rangle$. Recall, from a previous exercise, that $\sigma_z\sigma_x|\phi_1\rangle = |\phi_0\rangle$ for the Pauli operators

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \text{ and } \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

(1) Show that the maximally entangled state can be written as

$$|\Omega_2\rangle = \frac{1}{\sqrt{2}} \left(|\phi_0\rangle \otimes |\phi_0\rangle + |\phi_1\rangle \otimes |\phi_1\rangle \right).$$

(2) Suppose that Alice and Bob share a maximally entangled state. Alice applies the unitary U_θ^{-1} to her share of the state and measures her system in the computational basis. What are the postmeasurement states of Bob's system conditioned on the two measurement outcomes?

(3) Suppose that Alice knows the angle $\theta \in [0, 2\pi)$, but Bob does not. Find a protocol using 1 bit of classical communication and a shared maximally entangled state $|\Omega_2\rangle$ so that Bob ends up with the state $|\phi_0\rangle = U_\theta|0\rangle$ in the end[1].

[1]This task could be achieved using the teleportation protocol, but this would need 2 bits of classical communication.