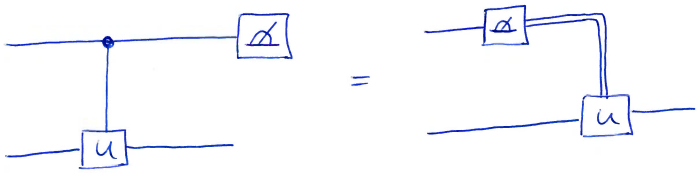
## EXERCISES 9

**Excercise 1** (The principle of deferred measurement)**.** Prove the following circuit identity involving a controlled (once quantumly controlled, and once classically controlled by the measurement outcome) unitary gate and a computational basis measurement:




Argue that all measurements at intermediate stages of a quantum algorithm can be moved to the end of the algorithm by replacing classically controlled unitaries by (quantumly) controlled unitaries.

**Excercise 2.** Let $H = \sum_{k=1}^{N} H_k$ denote a sum of Hermitian operators on $\mathbb{C}^d$ for some $d \in \mathbb{N}$. Assuming that $H_k$ and $H_l$ commute whenever $k \neq l$ show that

$$\exp(-iHt) = \exp(-iH_1t)\exp(-iH_2t)\cdots\exp(-iH_Nt),$$

for any $t \in \mathbb{R}$.

**Excercise 3** (The swap test)**.** Consider the operators

$$P_0 = \frac{1}{2}\left(\mathbb{1}_2 + \mathbb{F}_2\right) \quad \text{and} \quad P_1 = \frac{1}{2}\left(\mathbb{1}_2 - \mathbb{F}_2\right),$$

acting on $\mathbb{C}^2 \otimes \mathbb{C}^2$, where $\mathbb{F}_2$ denotes the swap (or flip) operator acting as $\mathbb{F}_2(|a\rangle \otimes |b\rangle) = |b\rangle \otimes |a\rangle$ for $|a\rangle, |b\rangle \in \mathbb{C}^2$.

(1) Show that $\{P_0, P_1\}$ defines a projective measurement.
(2) Compute the probabilities of observing outcome 0 and 1 when measuring a 2-qubit product state $|\psi\rangle \otimes |\phi\rangle$. Explain how to use this measurement to check whether two quantum states are equal (this is the so-called swap test).
(3) Find a quantum circuit to implement the measurement $\{P_0, P_1\}$ using only gates from the universal set $\{T, H, CNOT\}$ and computational basis measurements.

**Excercise 4.** For $N \in \mathbb{N}$ consider the matrix representation of the length $N$ *discrete Fourier transform* (DFT) given by

$$F_N = \frac{1}{\sqrt{N}} \begin{pmatrix} 1 & 1 & 1 & \cdots & 1 \\ 1 & \omega_N & \omega_N^2 & \cdots & \omega_N^{N-1} \\ 1 & \omega_N^2 & \omega_N^4 & \cdots & \omega_N^{2(N-1)} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & \omega_N^{N-1} & \omega_N^{2(N-1)} & \cdots & \omega_N^{(N-2)(N-1)} \end{pmatrix},$$

where $\omega_N = \exp(-2\pi i/N)$.

(1) Check that $F_N$ is a unitary matrix for any $N \in \mathbb{N}$.

(2) Let $N$ be even. Verify that

$$F_N = \begin{pmatrix} \mathbb{1}_{N/2} & D_{N/2} \\ \mathbb{1}_{N/2} & -D_{N/2} \end{pmatrix} \begin{pmatrix} F_{N/2} & \\ & F_{N/2} \end{pmatrix} P,$$

for some permutation matrix $P$, and where $D_{N/2}$ denotes the diagonal matrix with diagonal entries $-\omega_N^{N/2}, -\omega_N^{N/2+1}, \cdots, -\omega_N^{N-1}$. For $N = 2^n$ we can iterate this decomposition and decompose the matrix $F_N$ into a product of matrices with very few non-zero entries.