

# UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT3420 — Quantum Computing

Day of examination: Wednesday, 3 June 2026

Examination hours: 15:00–19:00

This problem set consists of 7 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

There are two parts to this exam. In Part I, answer all questions. These are multiple-choice questions, and you can just state your answer without justification. In Part II, justify your answer with full reasoning. Choose two out of the four problems in this part. You may answer more than two problems, but only the best two answers will count toward your score.

You can use these formulas without justification. The inner product on the Hilbert space  $\mathbb{C}^n$  is given by

$$\langle v, w \rangle = \sum_{i=1}^n v_i^* w_i;$$

The matrix product of an  $l \times m$  matrix  $A$  and an  $m \times n$  matrix  $B$  is given by

$$AB = \left[ \sum_{k=1}^m A_{ik} B_{kj} \right]_{i,j};$$

Qubit states:

$$|0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad |1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad |+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle);$$

Single qubit gates:

$$X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$
$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

(Continued on page 2.)

## Part I

### Problem 1

Which of the following is the most appropriate description of quantum mechanical states?

1. They are represented by unit vectors in a Hilbert space.
2. They are represented by arbitrary vectors in a Hilbert space.
3. They are represented by points in the phase space, where the Lagrangian is defined as a function giving differential equation to predict the time development.

### Problem 2

Suppose we have a qubit system in the state  $|+\rangle$ . After we perform a measurement in the computational basis, what is the most appropriate description of the system?

1. It will be in the state  $|+\rangle$ .
2. Depending on the outcome of the measurement, it will be in the state  $|0\rangle$  or  $|1\rangle$ .
3. It will be in the state  $|i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$ .

### Problem 3

Suppose we want to measure a qubit state using the basis  $(|+\rangle, |-\rangle)$ . Which set of matrices could be used as the projection-valued measure for this task?

1. the identity matrix  $I_2$  and the zero matrix  $0_2$ .
2. the matrices

$$\frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad \frac{1}{2} \begin{bmatrix} 1 & -1 \\ -1 & 1 \end{bmatrix}.$$

3. the unitary matrices  $X$  and  $Z$ .

### Problem 4

Suppose we apply the quantum gate  $H$  to a single-qubit system, and then apply the gate  $Y$ . Which matrix represents the overall composite transformation?

$$A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad B = \frac{1}{2} \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}, \quad C = \frac{1}{\sqrt{2}} \begin{bmatrix} -i & i \\ i & i \end{bmatrix}.$$

### Problem 5

Suppose we have the qubit state

$$|i\rangle = \frac{1}{\sqrt{2}}|0\rangle + \frac{i}{\sqrt{2}}|1\rangle.$$

(Continued on page 3.)

What is the probability  $\mathbb{P}[+]$  of obtaining the outcome  $|+\rangle$ ?

1.  $\mathbb{P}[+] = 0$ ,
2.  $\mathbb{P}[+] = \frac{1+i}{2}$ ,
3.  $\mathbb{P}[+] = \frac{1}{2}$ .

### Problem 6

Which of the following matrices is valid as a transformation of a single-qubit system?

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} i & -1 \\ 1 & -i \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

### Problem 7

Which of the following statements about the dimension of Hilbert spaces representing multi-qubit systems is correct?

1. Every time we increase the number of qubits, we need to increase the dimension by 1.
2. Every time we increase the number of qubits, we need to increase the dimension by 2.
3. Every time we increase the number of qubits, we need to double the dimension.

### Problem 8

Consider the following two-qubit states. Which of these is entangled?

$$v_1 = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \quad v_2 = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle), \quad v_3 = |01\rangle.$$

### Problem 9

Which of the following is the density matrix representing a mixture of states  $|+\rangle$  and  $|-\rangle$ , such that  $|+\rangle$  will be observed with probability  $\frac{1}{2}$ ?

$$A = \begin{bmatrix} \frac{1}{2} & -\frac{1}{2} \\ 0 & 0 \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & \frac{1}{2} \end{bmatrix}, \quad C = \begin{bmatrix} \frac{1}{2} & 0 \\ 0 & -\frac{1}{2} \end{bmatrix}.$$

### Problem 10

Which of the following explains quantum universality?

1. Starting from an appropriate choice of basic quantum gates, we can approximate arbitrary quantum gates to arbitrary precision by composition and tensor product.
2. Quantum gates are either reflections or rotations.
3. Quantum computation can be achieved by various combinations of NAND gates.

(Continued on page 4.)

## Part II

### Problem 11

Recall the CHSH game, involving two cooperating players, Alice and Bob, and a referee, Charlie:

- Charlie gives a question bit  $x = 0, 1$  to Alice, and another bit  $y = 0, 1$  to Bob.
- Alice answers with one bit  $a = 0, 1$ , and Bob similarly answers with one bit  $b = 0, 1$  (without communication).
- they win if  $a + b = xy \bmod 2$  holds, and lose otherwise.

The quantum algorithm is as follows:

1. Alice and Bob prepare the EPR pair state  $u = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$  in the two-qubit system. Each has access to one qubit.
2. Given the question  $x$ , Alice computes  $a = 0, 1$  as follows:
  - If  $x = 0$ , then she measures her part of the state in the computational basis. The outcome will be  $a$ .
  - If  $x = 1$ , then she measures her part in the basis  $(|+\rangle, |-\rangle)$ . The case “+” corresponds to  $a = 0$ , while the case “−” corresponds to  $a = 1$ .
3. Given the question  $y$ , Bob computes  $b = 0, 1$  as follows:
  - If  $y = 0$ , then he measures his part of the state in the basis

$$v_0 = \cos \frac{\pi}{8} |0\rangle + \sin \frac{\pi}{8} |1\rangle, \quad v_1 = \cos \frac{5\pi}{8} |0\rangle + \sin \frac{5\pi}{8} |1\rangle.$$

The case “ $v_0$ ” corresponds to  $b = 0$ , while the case “ $v_1$ ” corresponds to  $b = 1$ .

- If  $y = 1$ , then he measures his part in the basis

$$w_0 = \cos \frac{-\pi}{8} |0\rangle + \sin \frac{-\pi}{8} |1\rangle, \quad w_1 = \cos \frac{3\pi}{8} |0\rangle + \sin \frac{3\pi}{8} |1\rangle.$$

The case “ $w_0$ ” corresponds to  $b = 0$ , while the case “ $w_1$ ” corresponds to  $b = 1$ .

In the following questions, we assume that Charlie chooses questions  $(x, y) \in \{0, 1\}^2$  uniformly at random.

**a**

Suppose they started with the separable state  $u' = |01\rangle$  instead of  $u$ , but otherwise followed through the same protocol. What is their winning probability?

**b**

Suppose they started with the entangled state  $u'' = \frac{1}{\sqrt{2}}(|0+\rangle + |1-\rangle)$  instead of  $u$ , but otherwise followed through the same algorithm. Compare their winning probability to the one from the original algorithm, and to the one from part **a**. (You do not have to give a precise number for the probability.)

(Continued on page 5.)

## Problem 12

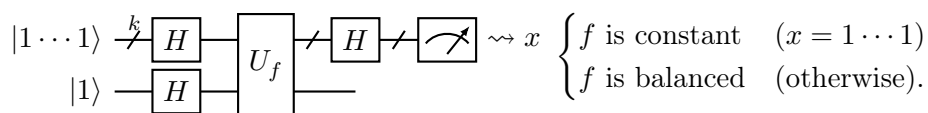
Recall the Deutsch–Josza problem:

We are given a function  $f(x)$  with input values  $x \in \{0, 1\}^k$  and output values  $f(x) \in \{0, 1\}$ , such that either  $f(x) = 0$  for exactly half of all inputs  $x$ , and  $f(x) = 1$  happens for the other half (the balanced case), or  $f(x) = i$  happens for all inputs  $x$ , for some constant  $i = 0, 1$  (the constant case). Decide which case holds.

Their quantum algorithm involving  $k + 1$  qubits uses one application of the XOR query gate,

$$U_f |xy\rangle = |xz\rangle \quad (x \in \{0, 1\}^k, y \in \{0, 1\}, z = \text{XOR}(y, f(x)))$$

as follows:



**a**

Suppose that the initial state preparation has an error, and we start with

$$((\cos \epsilon) |1 \cdots 1\rangle + (\sin \epsilon) v) \otimes |1\rangle$$

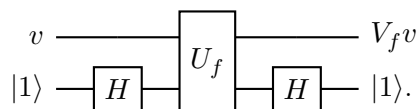
instead of  $|1 \cdots 1\rangle$ , where  $v$  is a vector orthogonal to  $|1 \cdots 1\rangle$ . Estimate the probability of getting a correct answer. We can assume that the quantum gates do not introduce further errors.

**b**

Recall the phase query gate,

$$V_f |x\rangle = (-1)^{f(x)} |x\rangle,$$

which satisfies



Design a  $k$ -qubit quantum circuit that solves the Deutsch–Josza problem with one use of the phase query gate.

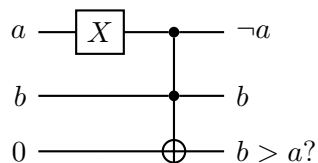
## Problem 13

We want to implement a reversible classical circuit that compares two 2-bit integers  $a$  and  $b$ , and flips a flag bit when  $a < b$  holds. Let us say that the 2-bit string  $a = a_0 a_1$  represents the integer  $2a_0 + a_1$ . The condition  $b > a$  can be expressed as

$$“b_0 > a_0” \text{ or } “b_0 = a_0 \text{ and } b_1 > a_1”.$$

(Continued on page 6.)

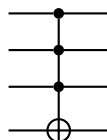
Comparison of single-bit inputs can be done by the Pauli  $X$ -gate followed by the Toffoli gate (double-controlled-NOT), as in:



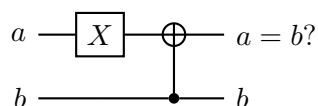
Here, we write  $a$  instead of  $|a\rangle$ , and “ $b > a$ ?” is 1 when  $b > a$  (that is,  $a = 0$  and  $b = 1$ ) holds.

**a**

Draw a reversible classical circuit with five wires, representing  $a_0, a_1, b_0, b_1$ , and  $t$  ( $t$  is the target bit), that flips  $t$  when “ $b_0 = a_0$  and  $b_1 > a_1$ ” holds. You can use the  $X$  (NOT) gate, the CNOT gate, and a multi-control CNOT gate like



Note that we can test bit-equality by



**b**

Draw the complete circuit with five wires as in part **a**, that receives  $a = a_0a_1$ ,  $b = b_0b_1$ , and  $f$ , and outputs  $a$  and  $b$  unchanged while flipping  $f$  when  $b > a$  holds.

If you did not manage part **a**, you can treat that part as a black box, but you should explain how to undo any changes to the  $a_i$  and the  $b_i$  bits by the gates mentioned above.

## Problem 14

Recall Grover’s search problem:

Find the “marked input”  $x^* \in \{0, 1\}^k$  specified by a Boolean function  $f: \{0, 1\}^k \rightarrow \{0, 1\}$ , where  $x^*$  is a solution for  $f(x^*) = 1$ .

His algorithm was given by measurement of

$$v_j = (DV_f)^j |+\cdots+\rangle$$

for an appropriate  $j$ , with the phase query operator

$$V_f |x\rangle = (-1)^{f(x)} |x\rangle$$

and the diffusion operator

$$D |x\rangle = \frac{1}{2^{k-1}} \sum_{y \in \{0,1\}^k} |y\rangle - |x\rangle.$$

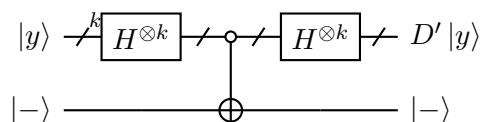
(Continued on page 7.)

**a**

In order to minimize error from quantum gates, we want to apply as few applications of  $D$  and  $V_f$  as possible to the system, as long as we can obtain a candidate for  $x^*$  with probability at least  $\frac{1}{2}$ . (We know that the marked input is unique.) Give an estimate for a good stopping value of  $j$ .

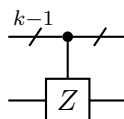
**b**

An implementation of the modified diffusion operator  $D' = -D$  with an auxiliary qubit was given by the following:



where, in the middle, we have the controlled-NOT gate with control condition  $x = 0 \cdots 0$  for  $k$ -bit  $x$ .

Give an implementation of  $D'$  without the auxiliary qubit. You can use the multi-controlled  $Z$  gate



that flips the sign of computational basis  $|11 \cdots 1\rangle$  while keeping the other states  $|x\rangle$  for  $x \in \{0, 1\}^k$ .

END