

MAT3420 (Quantum Computing) model answers

June 17, 2026

Part I

Answers and short explanations:

- 1: 1. (2. is wrong because we don't consider non-unit vectors as states, 3. is wrong because this description is about classical mechanics.)
- 2: 2. (1. and 3. are wrong because the state will collapse to one of the basis vectors used in the measurement.)
- 3: 2. (1. is wrong because this measurement scheme will always give the case " I_2 ", 3. is wrong because these matrices are not projections.)
- 4: C. (the answer should match the unitary gate YH . The matrices A and B are not invertible.)
- 5: 3. (the probability is given by $|\langle +|i\rangle|^2$, which is equal to $\frac{1}{2}$.)
- 6: A. (A transform of single-qubit system should be given by a 2×2 -unitary matrix. The matrices B and C are not unitary, because they don't have orthogonal columns and rows.)
- 7: 3. (The k -qubit system $(\mathbb{C}^2)^{\otimes k}$ has dimension 2^k . If we increase k by 1, this dimension becomes double.)
- 8: v_1 . (v_2 is the separable state $|++\rangle = |+\rangle \otimes |+\rangle$, as well as $v_3 = |0\rangle \otimes |1\rangle$.)
- 9: B. (It should be a density matrix. A is not positive semi-definite because it is not symmetric, and C is not positive definite because $\langle 1|C|1\rangle = -\frac{1}{2}$ is not non-negative.)
- 10: 1. (2. is wrong because higher-dimensional quantum gates are in general not either of these, and 3. is wrong because this explains the classical universality for the NAND gate.)

Part II

Problem 11

Problem 11a (12 points)

Since they start with the separable state $u' = |0\rangle \otimes |1\rangle$, there will be no correlation between the outcomes of Alice and Bob. Bob will compute his outcome b from his state $|1\rangle$ and the choice of basis determined by the question bit y .

When Alice receives $x = 0$ (this happens 50% of the time), her outcome will be always "0". In this case, Bob's outcome should be $b = 0$ for them to win. The configuration of his measurement bases and the state $|1\rangle$ is as in Figure 1. We see that they will more likely to lose, with probability $\cos^2 \frac{\pi}{8}$, and win with probability $\sin^2 \frac{\pi}{8}$.

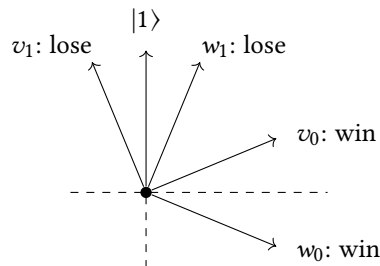


Figure 1: Bob's bases and win-lose cases under $x = 0$

When Alice receives $x = 1$ (this happens 50% of the time), she will use the $|\pm\rangle$ basis and will observe both cases with equal probability, so $a = 0$ and $a = 1$ are equally likely. In this case, if Bob receives $y = 0$ then the

winning condition is $b = a$, while if $y = 1$ it will be $b \neq a$. Then they would win with probability 50%.

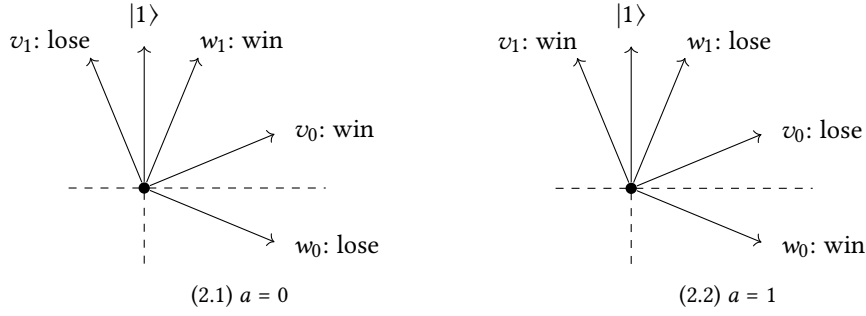


Figure 2: Bob's bases and win-lose cases under $x = 1$

Overall, their winning probability is

$$\frac{1}{2} \left(\sin^2 \frac{\pi}{8} + \frac{1}{2} \right).$$

Problem 11b (13 points)

When Alice gets $x = 0$, she will use the computational basis for measurement. If she observes the case 0 (which happens with probability 50%), Bob will have the state $|+\rangle$. If she observes the case 1 (which happens with probability 50%), he will have the state $|-\rangle$.

When $a = 0$, the configuration of Bob's bases and win-lose cases are as in Figure 3.1. If he gets $y = 0$, they are more likely to win, with probability $\cos^2 \frac{\pi}{8}$. If he gets $y = 1$, they are less likely to win, with probability $\sin^2 \frac{\pi}{8}$. Overall, their winning probability under $x = 0$ and $a = 0$ will be

$$\frac{1}{2} \left(\cos^2 \frac{\pi}{8} + \sin^2 \frac{\pi}{8} \right) = \frac{1}{2}.$$

The case $x = 0$ and $a = 1$ will lead to the same estimate as in Figure 3.2, and they win with probability $\frac{1}{2}$.

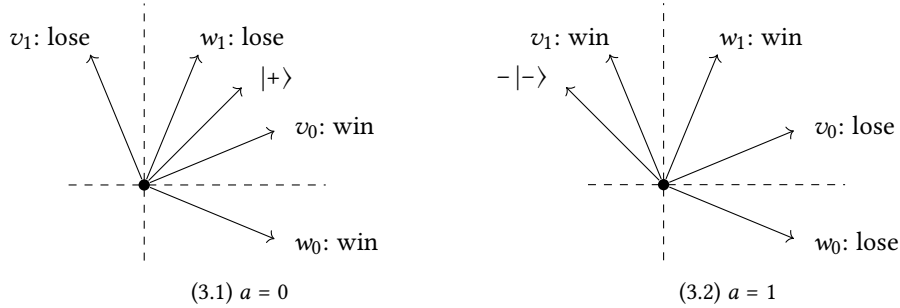


Figure 3: Bob's bases and win-lose cases under $x = 0$

Next let's see what happens after she gets $x = 1$. The state u'' can be also written as

$$u'' = \frac{1}{\sqrt{2}}(|+0\rangle + |-1\rangle).$$

To check this, we expand both formulas for u'' in terms of the computational basis to get

$$u'' = \frac{1}{2}(|00\rangle + |10\rangle + |01\rangle - |11\rangle).$$

This means, if her answer is $a = 0$ (the case $+$, which happens with probability 50%) Bob will have the state $|0\rangle$, and otherwise (which happens with probability 50%) he will have the state $|1\rangle$. If $x = 1$ and $a = 0$, the configuration of Bob's bases and win lose cases are as in Figure 4.1. By an similar estimate as above they win with probability $\frac{1}{2}$ under $x = 1$.

Overall, their winning probability is $\frac{1}{2}$. This is bigger than the winning probability from part a, but smaller than the one from original algorithm, which is around 85%.

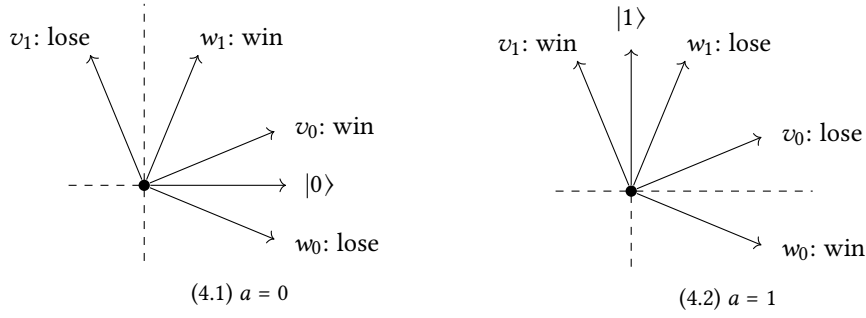


Figure 4: Bob's bases and win-lose cases under $x = 1$

Problem 12

Problem 12a (12 points)

When u is a initial $(k + 1)$ -qubit state vector, the circuit (before doing measurement) gives the state vector

$$u' = (H^{\otimes k} \otimes I_2)U_f H^{\otimes k+1}u.$$

The probability of observing $1 \dots 1$ in the first k qubits in this state is

$$\langle (|1 \dots 1\rangle \langle 1 \dots 1| \otimes I_2)u', u' \rangle = \langle (\langle 1 \dots 1| \otimes I_2)u', (\langle 1 \dots 1| \otimes I_2)u' \rangle.$$

Now, suppose that the initial state is

$$u = \cos \epsilon |1 \dots 1\rangle + \sin \epsilon v \otimes |1\rangle,$$

with v orthogonal to $|1 \dots 1\rangle$. The first part of circuit, $H^{\otimes k+1}$ transforms this state to

$$\cos \epsilon |-\dots-\rangle + \sin \epsilon v' \otimes |-\rangle,$$

with v' orthogonal to $|-\dots-\rangle$.

If f is constant, U_f preserves $|-\dots-\rangle$ up to sign $\pm$. On $v' \otimes |-\rangle$, the effect is $V_f(v') \otimes |-\rangle$ for the phase query gate V_f . Overall, the state will be

$$\pm \cos \epsilon |-\dots-\rangle + \sin \epsilon v'' \otimes |-\rangle,$$

with v'' orthogonal to $|-\dots-\rangle$. By taking the effect of $\langle 1 \dots 1| \otimes I_2$, this vector becomes $\pm \cos \epsilon |-\rangle$. This means that we can conclude that f is constant with probability at least $\cos^2 \epsilon$.

If f is balanced, then $U_f |-\dots-\rangle$ will be in the kernel of

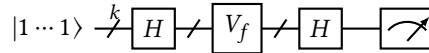
$$\langle -\dots- | \otimes I_2 = (\langle 1 \dots 1| \otimes I_2)(H^{\otimes k} \otimes I_2).$$

Then the value of $\langle (\langle 1 \dots 1| \otimes I_2)u', (\langle 1 \dots 1| \otimes I_2)u' \rangle$ will be at most $\sin^2 \epsilon$. This means we have at most $\sin^2 \epsilon$ of (wrongly) concluding that f is constant.

Overall, the probability of getting the correct answer is $\cos^2 \epsilon$.

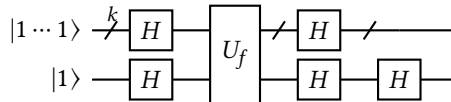
Problem 12b (13 points)

We use the circuit



If the measurement outcome is $1 \dots 1$, conclude that f is constant, and otherwise conclude that f is balanced.

The circuit for the DJ algorithm can be written as

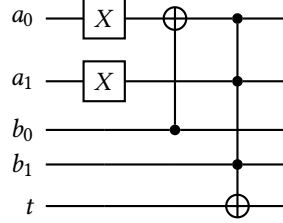


Using the presentation of V_f from problem, this is the same thing as $H^{\otimes k+1}$ acting on the state $(V_f |-\dots-\rangle) \otimes |1\rangle$. The final measurement is for the top k qubits, so we can ignore the last factor $|1\rangle$. In other words, measurement of $H^{\otimes k} V_f |-\dots-\rangle$ in the computational basis will agree with the one from the DJ algorithm.

Problem 13

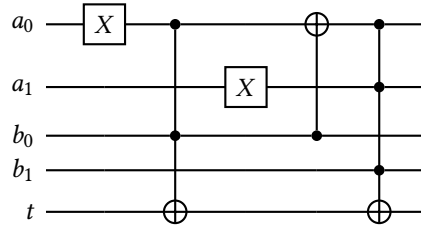
Problem 13a (12 points)

To flip the target bit when both " $a_0 = b_0$ " and " $a_1 < b_1$ " happens, we should use a multi-control CNOT whose control bits represent these conditions. The circuit for $a_1 < b_1$ is given in the problem description using the X-gate and the Toffoli gate. So we want to add the bit equality $a_0 = b_0$ as an extra control to the Toffoli gate. The combined circuit is:

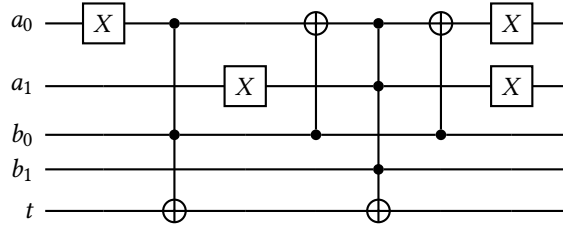


Problem 13b (13 points)

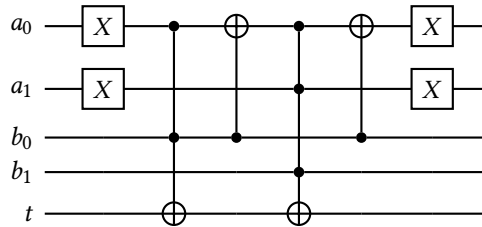
The conditions " $b_0 > a_0$ " and " $b_0 = a_0$ and $b_1 > a_1$ " are mutually exclusive. To flip the target bit under these conditions, we can just chain the corresponding circuit. After the first part checking $b_0 > a_0$, the top wire will represent $\neg a_0$, so we can omit the X-gate for that in part a. We then get



Next we have to cancel the effect of this circuit on the bits a_i and b_i . The effects come from the X-gate and the CNOT gate. Both are inverse of themselves, so we should start with applying CNOT on b_0 and a_0 , then X on a_0 and a_1 . The completed circuit will be



or equivalently



Problem 14

Problem 14a (12 points)

The transforms D and V_f preserves the 2-dimensional span of $|+\dots+\rangle$ and $|x^*\rangle$. Each application of DV_f will rotate the state from $|+\dots+\rangle$ towards $|x^*\rangle$ by angle 2θ in this 2-dimensional space, where θ is characterized by $\sin \theta = \frac{1}{\sqrt{2^k}}$.

To measure x^* with probability close to $\frac{1}{2}$, the inner product $\langle x^* | (DV_f)^j | + \dots + \rangle$ should be close to its square root, $\frac{1}{\sqrt{2}}$. Since we start from the state $| + \dots + \rangle$ which is almost orthogonal to $| x^* \rangle$, we want rotation of angle close to $\frac{\pi}{4}$ to achieve this. The stop count j should satisfy $2j\theta \sim \frac{\pi}{4}$, hence $j \sim \frac{\pi\sqrt{2^k}}{8}$.

Problem 14b (13 points)

We have

$$D'v = \sum_{x \in \{0,1\}^x} (\alpha_x - 2\bar{\alpha}) |x\rangle \quad (v = \sum_x \alpha_x |x\rangle),$$

where $\bar{\alpha}$ denotes the average of the coefficients $(\alpha_x)_x$. This implies

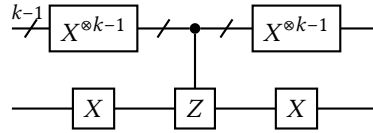
$$D'(| + \dots + \rangle) = -| + \dots + \rangle, \quad D'(|y_1 \dots y_k\rangle) = |y_1 \dots y_k\rangle \quad (y = y_1 \dots y_k \in \{+, -\}^k, y \neq + \dots +).$$

since $\bar{\alpha} = 1$ and $\alpha_x = 1$ for all x in the first case, while $\bar{\alpha} = 0$ in the second case.

By conjugating by $H^{\otimes k}$, this becomes

$$|0 \dots 0\rangle \mapsto -|0 \dots 0\rangle, \quad |x_1 \dots x_k\rangle \mapsto |x_1 \dots x_k\rangle \quad (x = x_1 \dots x_k \in \{0, 1\}^k, x \neq 0 \dots 0).$$

This is given by



Combining conjugation by H , the desired circuit becomes

