

QUANTUM COMPUTATION: EXERCISE SOLUTIONS

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1. FEBRUARY 5

Suppose we have two matrices $A = [A_{ij}]_{i,j}$ and $B = [B_{kl}]_{k,l}$ with $i = 1, \dots, m$, $j = 1, \dots, n$, $k = 1, \dots, p$, and $l = 1, \dots, q$. Their *Kronecker product* is

$$\begin{bmatrix} A_{11}B & A_{12}B & \dots & A_{1n}B \\ \vdots & \vdots & \dots & \vdots \\ A_{m1}B & A_{m2}B & \dots & A_{mn}B \end{bmatrix}.$$

(This is a block matrix notation, so $A_{11}B$ represents a block of p rows and q columns with components $A_{11}B_{kl}$, and so on, and overall there are mp rows and nq columns.) We can think of this as a concrete matrix representation of the tensor product $A \otimes B$. For vectors $v \in \mathbb{C}^n$ and $w \in \mathbb{C}^q$, we think of them as $n \times 1$ -matrix and $q \times 1$ -matrix, and use the Kronecker product to represent $v \otimes w$.

Exercise 1.1. In the above setting, check $(A \otimes B)(v \otimes w) = Av \otimes Bw$. You can start with $m = n = p = q = 2$ to get the feeling.

Solution. See computational note file (`mat3420-v26.ipynb`) for the $m = n = p = q = 2$ case. $\square$

Exercise 1.2 (NC, Exercise 2.26). Consider $|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$. Write $|++\rangle = |+\rangle \otimes |+\rangle$ as a linear combination of vectors $|ij\rangle = |i\rangle \otimes |j\rangle$ with $i, j = 0, 1$. Next, represent $|++\rangle$ by the Kronecker product of column vectors.

Solution. We get $|++\rangle = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$. See also the computational note file. $\square$

Consider the *Hadamard gate* operator on $\mathbb{C}^2$, given as

$$H = \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}.$$

Exercise 1.3 (NC, Exercise 2.51). Check that H is a unitary matrix. Figure out how H transforms the vectors $|0\rangle$, $|1\rangle$, $|+\rangle$, $|-\rangle$, $|i\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle)$, and $|-i\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle)$ (i.e., the last vector is the eigenvector of the Pauli Y matrix for eigenvalue $-i$, and not the tensor product vector $|-\rangle \otimes |i\rangle$).

Solution. We get

$$H|0\rangle = |+\rangle, \quad H|1\rangle = |-\rangle, \quad H|+\rangle = |0\rangle, \quad H|-\rangle = |1\rangle, \quad H|i\rangle = \frac{1+i}{\sqrt{2}}|-i\rangle, \quad H|-i\rangle = \frac{1-i}{\sqrt{2}}|i\rangle,$$

see the computational note. You can use $H^2 = I_2$, which implies $Hv = w \Leftrightarrow v = Hw$, to get $H|+\rangle = |0\rangle$ from $H|0\rangle = |+\rangle$, etc. $\square$

Exercise 1.4. What kind of rotation does H define on the Bloch sphere?

Solution. We know how the marked points $|x\rangle$ ($x = 0, 1, +, -, i, -i$) on the Bloch sphere are moved by the effect of H . The corresponding rotation is the 180° rotation through the dashed axis in Figure 1.1. This axis goes through the point u_1 halfway between $|0\rangle$ and $|1\rangle$, and the point u_2 halfway between $|1\rangle$ and $|-\rangle$. $\square$

Exercise 1.5. Relate the axis of rotation in the previous exercise to the eigenvectors of H .

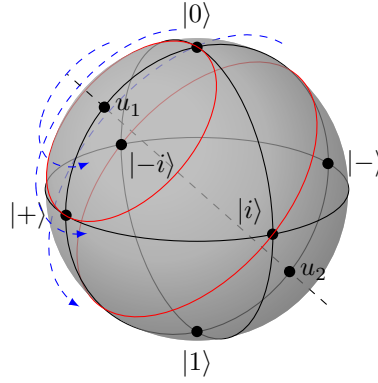


FIGURE 1.1. Hadamard transform on the Bloch sphere

Solution. The eigenvectors of H are given by

$$\cos\left(\frac{\pi}{8}\right) |0\rangle + \sin\left(\frac{\pi}{8}\right) |1\rangle, \quad \cos\left(\frac{5\pi}{8}\right) |0\rangle + \sin\left(\frac{5\pi}{8}\right) |1\rangle,$$

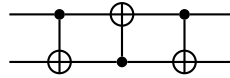
with eigenvalues 1 and -1 , see the computational note. The angles come from the observation

$$|+\rangle = \cos\left(\frac{\pi}{4}\right) |0\rangle + \sin\left(\frac{\pi}{4}\right) |1\rangle, \quad |+\rangle = \cos\left(\frac{3\pi}{4}\right) |0\rangle + \sin\left(\frac{3\pi}{4}\right) |1\rangle.$$

Be careful that these angles are half of the corresponding angles on the Bloch sphere. $\square$

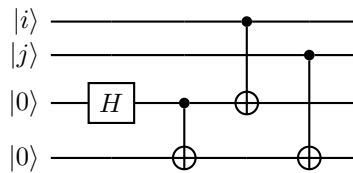
2. FEBRUARY 26

Exercise 2.1. What does this two-qubit quantum circuit do to separable states?



Solution. This circuit acts as flip on tensor product factors, i.e., transforms $v \otimes w$ to $w \otimes v$ for $v, w \in \mathbb{C}^2$. $\square$

Exercise 2.2. Consider the following quantum circuit, with top two input qubits $|i\rangle, |j\rangle$ in computational basis. A certain measurement scheme of the bottom two qubits lets us decide $i = j$ or not. What is such a scheme?



(Hint: the Hadamard gate and the first CNOT will create the EPR pair state.)

Solution. We should check if the bottom two qubits combined is the EPR state or not. More formally, we use the PVM

$$E_{=} = w_{\text{EPR}} w_{\text{EPR}}^\dagger, \quad E_{\neq} = I_4 - E_{=}$$

on the bottom two qubits. This works because the cases $i = 0 = j$ and $i = 1 = j$ would preserve the EPR pair state w_{EPR} in the bottom two qubits, while $i = 0, j = 1$ would change it to $\frac{1}{\sqrt{2}}(|01\rangle + |10\rangle)$, which is orthogonal to w_{EPR} , and the same happens for $i = 1, j = 0$. $\square$

Exercise 2.3. Consider the following states we had in the superdense coding protocol:

$$w_{00} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle), \quad w_{10} = (X \otimes I_2)(w_{00}), \quad w_{01} = (Z \otimes I_2)(w_{00}), \quad w_{11} = (ZX \otimes I_2)(w_{00}).$$

How do they look like in the Kronecker product presentation?

Solution. They look like

$$w_{00} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ \frac{1}{\sqrt{2}} \end{bmatrix}, \quad w_{10} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}, \quad w_{01} = \begin{bmatrix} \frac{1}{\sqrt{2}} \\ 0 \\ 0 \\ -\frac{1}{\sqrt{2}} \end{bmatrix}, \quad w_{11} = \begin{bmatrix} 0 \\ \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{2}} \\ 0 \end{bmatrix}. \quad \square$$

Exercise 2.4. Suppose we have a vector $w \in \mathbb{C}^2 \otimes \mathbb{C}^2$. We can consider the transform

$$T_w: \mathbb{C}^2 \rightarrow \mathbb{C}^2, \quad v \mapsto (v^\dagger \otimes I_2)(w).$$

(This is not a linear map for the complex coefficient, but instead a *conjugate linear* map.) Check the following:

- (1) when $w = w_1 \otimes w_2$, the image of T_w , i.e., $\{T_w(v) \mid v \in \mathbb{C}^2\}$ is a one-dimensional subspace of $\mathbb{C}^2$.
- (2) when $w = |00\rangle + |11\rangle$, the image of T_w is all of $\mathbb{C}^2$.

Solution. When $w = w_1 \otimes w_2$, we compute

$$T_w(v) = v^\dagger w_1 \otimes I_2 w_2 = \langle v, w_1 \rangle w_2.$$

This means that the range of T_w is the 1-dimensional span of v .

When $w = |00\rangle + |11\rangle$, we compute

$$T_w(v) = v^\dagger |0\rangle \otimes I_2 |0\rangle + v^\dagger |1\rangle \otimes I_2 |1\rangle = v_1^* |0\rangle + v_2^* |1\rangle.$$

Then any vector $u \in \mathbb{C}^2$ can be written as $T_w(v)$: we set $v_1 = u_1^*$ and $v_2 = u_2^*$. $\square$

3. MARCH 24

Exercise 3.1. Consider the Boolean function $f: \{0, 1\}^2 \rightarrow \{0, 1\}$ defined by $f(x_1 x_2) = x_1 + x_2 \bmod 2$. We then have the XOR query gate U_f on the 3-qubit system defined by

$$U_f |x_1 x_2 y\rangle = |x_1 x_2 z\rangle, \quad z = \text{XOR}(f(x_1 x_2), y)$$

on the computational basis. Why is this a unitary transform? How about the same construction for $f(x_1 x_2) = x_1 \cdot x_2$?

Hint: Compute U_f^2 to find the inverse of U_f . Can you check that $U_f = U_f^\dagger$? You can either use an abstract reasoning, or work out the 8×8 -matrix representation of U_f .

Solution (abstract). Let us first check $U_f^2 = I_2^{\otimes 3}$. The effect of U_f^2 on computational basis is given by

$$U_f^2 |xy\rangle = U_f(U_f(|xy\rangle)) = U_f(|xz\rangle)$$

with z related to y as above. Using the same rule again, we get

$$U_f(|xz\rangle) = |xw\rangle,$$

where $w = 0, 1$ is given by $w = \text{XOR}(f(x_1 x_2), z)$. We have

$$\text{XOR}(a, \text{XOR}(a, b)) = b \quad (a, b = 0, 1)$$

from direct verification or $\text{XOR}(a, b) = a + b \bmod 2$. Then $a = f(x_1 x_2)$ and $b = y$ gives $w = y$, hence $U_f^2 |xy\rangle = |xy\rangle$ for any $x = x_1 x_2$ and y . This shows $U_f^2 = I_2^{\otimes 3}$.

Next let us check that U_f is self-adjoint. We have

$$\langle x' y' | U_f |xy\rangle = \begin{cases} 1 & x' = x \text{ and } y' = \text{XOR}(f(x), y) \\ 0 & \text{otherwise} \end{cases}$$

for $x, x' \in \{0, 1\}^2$ and $y, y' \in \{0, 1\}$. On the other hand,

$$\langle U_f |x' y'\rangle, |xy\rangle = \begin{cases} 1 & x' = x \text{ and } y = \text{XOR}(f(x), y') \\ 0 & \text{otherwise.} \end{cases}$$

We have $y' = \text{XOR}(f(x), y)$ if and only if $y = \text{XOR}(f(x), y')$. This implies

$$\langle U_f |x' y'\rangle, |xy\rangle = \langle x' y' | U_f |xy\rangle,$$

which means $U_f = U_f^\dagger$.

Now we got $U_f^\dagger = U_f = U_f^{-1}$, hence U_f is unitary. $\square$

Solution (concrete). Working out the effect of U_f , we get

$$\begin{aligned} |000\rangle &\mapsto |000\rangle, & |001\rangle &\mapsto |001\rangle, & |010\rangle &\mapsto |011\rangle, & |011\rangle &\mapsto |010\rangle, \\ |100\rangle &\mapsto |101\rangle, & |101\rangle &\mapsto |100\rangle, & |110\rangle &\mapsto |110\rangle, & |111\rangle &\mapsto |111\rangle. \end{aligned}$$

In the table format, this can be represented as in Table 1.

TABLE 1. Input-output table for the XOR query gate.

input \ output	000	001	010	011	100	101	110	111
000	*							
001		*						
010				*				
011			*					
100						*		
101					*			
110							*	
111								*

In the matrix format, this would be

$$U = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix},$$

which satisfies $U^2 = I_8$ and $U^\dagger = U$. □

Exercise 3.2. Given a Boolean function $f: \{0,1\}^k \rightarrow \{0,1\}^k$, recall the XOR query gate from Simon's algorithm:

$$U_f |xy\rangle = |xz\rangle \quad z = f(x) \oplus y$$

with the bit-wise XOR of $f(x)$ and y . Why is this a unitary transform?

Solution. On the one hand, the bit-wise XOR still satisfies $f(x) \oplus f(x) \oplus y = y$ by the same reason as above. This gives $U_f^2 = I$.

On the other hand, we can also check $U_f^\dagger = U_f$ in a same way as before. To be precise, we have

$$\langle x'y' | U_f |xy\rangle = \begin{cases} 1 & x' = x \text{ and } y' = f(x) \oplus y \\ 0 & \text{otherwise} \end{cases}$$

for $x, x' \in \{0,1\}^k$ and $y, y' \in \{0,1\}^k$. On the other hand,

$$\langle U_f |x'y'\rangle, |xy\rangle = \begin{cases} 1 & x' = x \text{ and } y = f(x) \oplus y' \\ 0 & \text{otherwise.} \end{cases}$$

We have $y' = f(x) \oplus y$ if and only if $y = f(x) \oplus y'$, hence $U_f^\dagger = U_f$. □

In the CHSH game (Lecture 12), Alice and Bob used different measurement schemes depending on the question bits x, y . (For example, if $x = 0$, Alice would measure her part of the state in the computational basis, while if $x = 1$ she would measure it in the $(|+\rangle, |-\rangle)$ -basis.)

Suppose they might choose a different set of basis for these questions, and we want to estimate their winning probability. Let's say Alice would use a basis (v_0, v'_0) of $\mathbb{C}^2$ for the case $x = 0$, and she would use a basis (v_1, v'_1) for the case $x = 1$. Set

$$A^0 = v_0 v_0^\dagger - v'_0 v_0'^\dagger, \quad A^1 = v_1 v_1^\dagger - v'_1 v_1'^\dagger.$$

For example, in the original CHSH strategy we would have

$$A^0 = |0\rangle\langle 0| - |1\rangle\langle 1| = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad A^1 = |+\rangle\langle +| - |-\rangle\langle -| = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

For Bob's choice of measurement schemes, we define B^0 and B^1 in the same way. Finally, let's say u is the two-qubit state they are going to use (which was the EPR pair state for the CHSH strategy).

Exercise 3.3. When the question is $x = 0$ and $y = 0$, check that their winning probability is

$$\left\langle u, \frac{1}{2}(A^0 \otimes B^0 + I_4)u \right\rangle.$$

Hint: A^0 has eigenvalues 1 and -1 , corresponding to the cases $a = 0$ and $a = 1$ for Alice's answer. Same kind of relation holds for B^0 with Bob's answer b . Then $A^0 \otimes B^0$ has eigenvalues 1 and -1 , corresponding to the different combinations of a and b . Taking $\frac{1}{2}(A^0 \otimes B^0 + I_4)$ instead of $A^0 \otimes B^0$ has the effect of changing the eigenvalues from ± 1 to 1, 0, while keeping the same conditions about a and b .

Solution. Let's say Bob would use a basis (w_0, w'_0) for the question $y = 0$. Then we have

$$\begin{aligned} A^0 \otimes B^0 &= (v_0 v_0^\dagger - v'_0 v_0'^\dagger) \otimes (w_0 w_0^\dagger - w'_0 w_0'^\dagger) \\ &= v_0 v_0^\dagger \otimes w_0 w_0^\dagger - v'_0 v_0'^\dagger \otimes w_0 w_0^\dagger - v_0 v_0^\dagger \otimes w'_0 w_0'^\dagger + v'_0 v_0'^\dagger \otimes w'_0 w_0'^\dagger. \end{aligned}$$

Eigenvectors are $v_0 \otimes w_0$ and $v'_0 \otimes w'_0$ for eigenvalue 1, and $v'_0 \otimes w_0$ and $v_0 \otimes w'_0$ for eigenvalue -1 .

As the hint says (or directly computing the effect) $\frac{1}{2}(A^0 \otimes B^0 + I_4)$ has eigenvectors $v_0 \otimes w_0$ and $v'_0 \otimes w'_0$ for eigenvalue 1, and $v'_0 \otimes w_0$ and $v_0 \otimes w'_0$ for eigenvalue 0, i.e., this is a projection. Then

$$\left\langle u, \frac{1}{2}(A^0 \otimes B^0 + I_4)u \right\rangle$$

picks up the probability of measuring the cases corresponding to the eigenvalue 1.

The case of eigenvalue 1 corresponds to $a = b = 0$ or $a = b = 1$, which satisfy $a + b = 0 = xy \bmod 2$, i.e., the winning case. The case of eigenvalue 0 corresponds to $a = 0$ and $b = 1$, or $a = 1$ and $b = 0$, which satisfy $a + b = 1 \neq 0 = xy \bmod 2$, i.e., the losing case. $\square$

Exercise 3.4. For other choices of questions x and y , find signs in

$$\left\langle u, \frac{1}{2}(\pm A^x \otimes B^y + I_4)u \right\rangle.$$

such that this would represent their winning probabilities.

Solution. If $x = 0$ or $y = 0$, the winning condition is still $a + b = 0 \bmod 2$. Then the sign configuration is the same as the case of $x = y = 0$. If $x = y = 1$, the winning condition becomes $a + b = 1 \bmod 2$. This corresponds to the states $v'_1 \otimes w_1$ and $v_1 \otimes w'_1$, and the corresponding projection is $\frac{1}{2}(-A^1 \otimes B^1 + I_4)$. Thus the winning probability is given by

$$\left\langle u, \frac{1}{2}(-A^1 \otimes B^1 + I_4)u \right\rangle. \quad \square$$

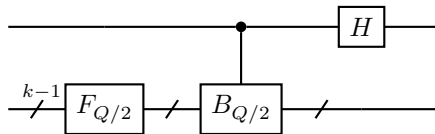
4. APRIL 23

Our convention of the Fourier transform matrix F_Q for $Q = 2^k$ is

$$[F_Q]_{j,l} = \frac{1}{\sqrt{Q}} \exp\left(\frac{2\pi i}{Q} j \tilde{l}\right),$$

where we modify the input index $0 \leq l < Q$ to another integer $0 \leq \tilde{l} < Q$ by the rule $\tilde{l} = 2^{k-1}x_0 + \dots + x_{k-1}$ if and only if $\tilde{l} = x_0 + 2x_1 + \dots + 2^{k-1}x_{k-1}$. (This means that the input binary sequence $x_0 \dots x_{k-1}$ in the quantum gate F_Q should be interpreted as the integer $\tilde{l}$.)

Exercise 4.1. Recall that F_Q can be represented by the quantum circuit

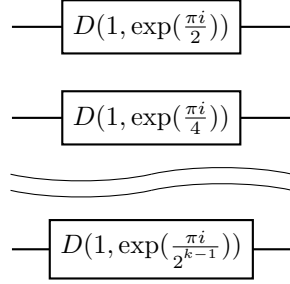


with $B_{Q/2}$ given by

$$B_{Q/2} = \begin{bmatrix} 1 & 0 & \cdots & \\ 0 & \exp\left(\frac{2\pi i}{Q}\right) & 0 & \cdots \\ \vdots & & \ddots & \\ 0 & \cdots & 0 & \exp\left((2^{k-1}-1)\frac{2\pi i}{Q}\right) \end{bmatrix}.$$

Check this for $k = 2$, i.e., check the effect of this circuit on the computational basis vectors $|x_0x_1\rangle$, and compare with the effect of F_4 .

Hint: We had $F_2 = H$ for $k = 1$, the Hadamard gate. As for $B_{Q/2}$, you can use the following $(k-1)$ -qubit quantum circuit representation:



(We didn't have time for this part during the lecture, but see the lecture 13 in the lecture note PDF.)

Solution. Given input 2-bit string $x = x_0x_1$, the corresponding integers $l = 2^{k-1}x_0 + \cdots + x_{k-1}$ (that has to do with Kronecker product convention and the matrix index) and $\tilde{l} = x_0 + 2x_1 + \cdots + 2^{k-1}x_{k-1}$ (that has to do with Fourier transform) are given by the following table.

TABLE 2. bit string and corresponding integers

x_0x_1	0 0	0 1	1 0	1 1
l	0	1	2	3
$\tilde{l}$	0	2	1	3

Then the Fourier transform matrix F_4 is given by

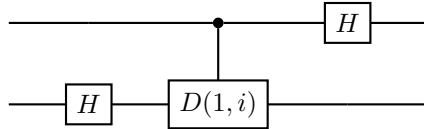
$$\frac{1}{\sqrt{4}} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & \exp(\frac{2\pi i}{4} \cdot 2) & \exp(\frac{2\pi i}{4}) & \exp(\frac{2\pi i}{4} \cdot 3) \\ 1 & \exp(\frac{2\pi i}{4} \cdot 4) & \exp(\frac{2\pi i}{4} \cdot 2) & \exp(\frac{2\pi i}{4} \cdot 6) \\ 1 & \exp(\frac{2\pi i}{4} \cdot 6) & \exp(\frac{2\pi i}{4} \cdot 3) & \exp(\frac{2\pi i}{4} \cdot 9) \end{bmatrix} = \frac{1}{2} \begin{bmatrix} 1 & 1 & 1 & 1 \\ 1 & -1 & i & -i \\ 1 & 1 & -1 & -1 \\ 1 & -1 & -i & i \end{bmatrix},$$

using $\exp(\pi i) = 1$, $\exp(\frac{\pi i}{2}) = i$, etc.

The matrix $B_{Q/2}$ for $Q = 4$ is

$$\begin{bmatrix} 1 & 0 \\ 0 & \exp(\frac{2\pi i}{4}) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix} = D(1, i).$$

Combined with $F_2 = H$, the quantum circuit for F_4 becomes



Let us consider the input string $x = 00$. First, H on the bottom string gives $H|0\rangle = |+\rangle$. Next, the control- $D(1, i)$ does not activate since we have 0 on the control bit. Finally, H on the top string gives $|+\rangle$ again, and the overall output is $|+\rangle \otimes |+\rangle = |++\rangle$. This is consistent with the first column (corresponding to $l = 0$) of F_4 .

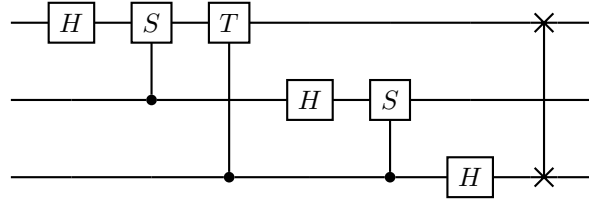
Next, let us consider the input string $x = 01$. First, H on the bottom string gives $H|1\rangle = |-\rangle$. Next, the control- $D(1, i)$ does not activate since we have 0 on the control bit. Finally, H on the top

string gives $|+\rangle$, and the overall output is $|+\rangle \otimes |-\rangle = |+-\rangle$. This is consistent with the second column (corresponding to $l = 1$) of F_4 .

Next, let us consider the input string $x = 10$. First, H on the bottom string gives $H|0\rangle = |+\rangle$. Next, the control- $D(1, i)$ activates since we have 1 on the control bit, and the bottom qubit becomes $\frac{1}{\sqrt{2}}(|0\rangle + i|1\rangle) = |i\rangle$. Finally, H on the top string gives $|-\rangle$, and the overall output is $|-\rangle \otimes |i\rangle = |-i\rangle$. This is consistent with the third column (corresponding to $l = 2$) of F_4 .

Finally, let us consider the input string $x = 11$. First, H on the bottom string gives $H|1\rangle = |-\rangle$. Next, the control- $D(1, i)$ activates since we have 1 on the control bit, and the bottom qubit becomes $\frac{1}{\sqrt{2}}(|0\rangle - i|1\rangle) = |i^*\rangle$. Finally, H on the top string gives $|-\rangle$, and the overall output is $|-\rangle \otimes |i^*\rangle = |-i^*\rangle$. This is consistent with the fourth column (corresponding to $l = 3$) of F_4 . $\square$

Exercise 4.2. We can find the following quantum circuit diagram for the 3-qubit Fourier transform in the Nielsen–Chuang book, Box 5.1.



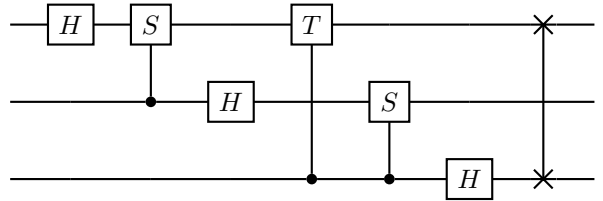
Here, the S -gate and T -gate are given by

$$S = \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad T = \begin{bmatrix} 1 & 0 \\ 0 & e^{i\pi/4} \end{bmatrix}.$$

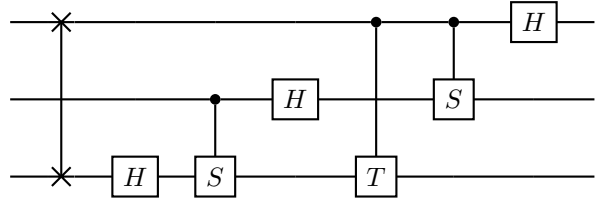
Can you relate this with the presentation from the previous exercise?

Hint: there are different conventions for the Fourier transform, like how bit-strings correspond to integers. First ignore the bit swap at the end of the above circuit.

Solution. The H gate on the middle qubit and the control- T gate on the top and bottom qubits are interchangeable, so we can rewrite the circuit as



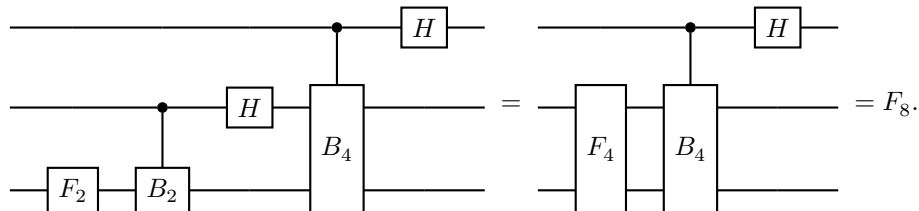
If we do the qubit flip at the beginning, we would get



We also have

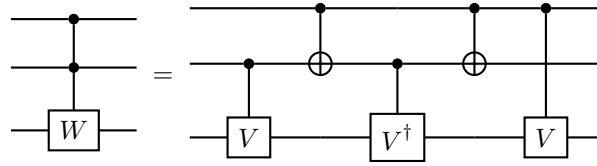
$$S = D(1, e^{i\pi/2}), \quad T = D(1, e^{i\pi/4}),$$

so the controlled gates can be interpreted as control- $B_{Q/2}$ for $Q = 4$ and $Q = 8$. Then, the parts other than the initial flip becomes



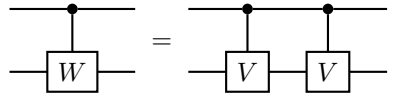
The bit flip just corresponds to different conventions of the Fourier transform, i.e., whether to use index $\tilde{l}$ or l in the definition. $\square$

Exercise 4.3. We saw that 2-bit control gates can be implemented as

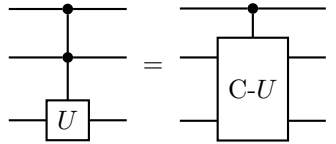


for a unitary matrix V satisfying $V^2 = W$. How about 3-bit control gates?

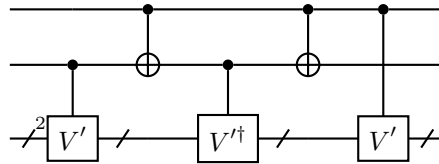
Hint: for the above formula it's not important that W is a single-qubit gate, and if we have $V^2 = W$, then we also have



Solution. When U is a quantum gate, let's write C- U for the controlled U -gate. Then the double controlled U -gate can be interpreted as the control- W gate for $W = \text{C-}U$,

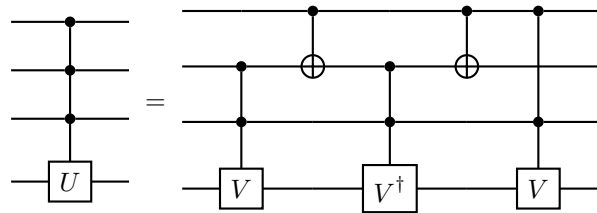


Then the triple controlled- U gate can be interpreted as double controlled-C- U . Using the hint, this can be written as



for a quantum gate V' satisfying $V'^2 = \text{C-}U$.

If a quantum gate V satisfies $V^2 = U$, we have $V'^2 = \text{C-}U$ for $V' = \text{C-}V$ as in the hint. This gives



$\square$