

QUANTUM COMPUTATION: MOCK EXAM

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We will freely use the following notation for qubit states:

$$|+\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad |-\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle),$$

and the following ones for single qubit gates:

$$\begin{aligned} X &= \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, & Y &= \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, & Z &= \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \\ S &= \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, & H &= \frac{1}{\sqrt{2}} \begin{bmatrix} 1 & 1 \\ 1 & -1 \end{bmatrix}. \end{aligned}$$

PART 1

Answer all questions from this part. These are all multiple-choice questions, and you can just state your answer without justification.

Problem 1. Which of the following is the most appropriate comparison between classical and quantum mechanics?

- (1) They are based on the same paradigm, but quantum mechanics involves different mathematical tools with systems at very small scale.
- (2) Measurement of a system in quantum mechanics changes it instantaneously and irreversibly, while classical mechanics has no such feature.
- (3) Classical mechanics involves differential equations given by Hamiltonian to capture phenomena at large scale, and quantum mechanics involves matrix equations to capture those at small scale.

Problem 2. Suppose we transform a qubit system by the unitary gate S , then perform another transform by the unitary gate X . Which matrix represents the overall composite transform?

$$A = \begin{bmatrix} 1 & 0 \\ 0 & -i \end{bmatrix}, \quad B = \begin{bmatrix} 0 & i \\ 1 & 0 \end{bmatrix}, \quad C = \begin{bmatrix} 0 & 1 \\ i & 0 \end{bmatrix}.$$

Problem 3. Suppose we want to measure a qubit state in the computational basis. Which set of matrices are the most relevant for this task?

- (1) the identity matrix I_2 and the zero-matrix 0_2 .
- (2) the diagonal matrices

$$\begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}.$$

- (3) the unitary matrices X and Z .

Problem 4. Suppose we had a qubit state, and after a measurement we can say that our state is $|+\rangle$. Which is the most appropriate description of the measurement?

- (1) Since $|-\rangle$ spans the orthogonal complement of $|+\rangle$ in the qubit system, the measurement corresponds to the choice $|+\rangle, |-\rangle$ as an orthonormal basis.
- (2) Since $|+\rangle$ represents an equal superposition of $|0\rangle$ and $|1\rangle$, the measurement corresponds to the choice of $|0\rangle, |1\rangle$ as an orthonormal basis.
- (3) None of the above

Problem 5. Suppose we have a qubit state

$$|\psi\rangle = \frac{4}{5}|0\rangle + \frac{3i}{5}|1\rangle.$$

What is the probability of observing $|+\rangle$?

- (1) $\frac{4+3i}{5\sqrt{2}}$
- (2) $\frac{1}{2}$
- (3) None of the above

Problem 6. Which of the following matrix is valid as a transform of qubit system?

$$A = \frac{1}{\sqrt{2}} \begin{bmatrix} i & -1 \\ 1 & -i \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{2}{3} & \frac{1}{3} \end{bmatrix}, \quad C = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}.$$

Problem 7. What goes in _ for the claim: “To model a system of _ qubits, we use a 4-dimensional Hilbert space.”?

- (1) 2
- (2) 4
- (3) None of the above

Problem 8. Which of the following is appropriate?

- (1) $v = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle)$ is an entangled state, because it has evenly distributed coefficients.
- (2) $w = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle - |11\rangle)$ is an entangled state, because the partial trace $(\text{id}_A \otimes \text{tr}_B)(\rho)$ of the corresponding density operator $\rho = ww^\dagger$ is not of rank 1.
- (3) None of the above

Problem 9. In this problem we work with a single qubit state space, and use the basis $|0\rangle, |1\rangle$ to represent matrices. Which of the following is the density matrix representing a superposition of states $|+\rangle$ and $|-\rangle$, such that $|+\rangle$ will be observed with probability $\frac{3}{4}$?

$$A = \begin{bmatrix} \frac{1}{2} & \frac{1}{4} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}, \quad B = \begin{bmatrix} \frac{1}{2} & \frac{1}{2} \\ \frac{1}{2} & -\frac{1}{2} \end{bmatrix}, \quad C = \begin{bmatrix} \frac{3}{4} & 0 \\ \frac{1}{4} & 0 \end{bmatrix}.$$

Problem 10. Suppose that we have a state

$$v = \frac{1}{2}(|00\rangle + |01\rangle - |10\rangle + |11\rangle)$$

in a two-qubit system, and partial measurement of the first qubit of this state, in the computational basis, gave the case “1” (corresponding to $|1\rangle$). How does the second qubit look like after this?

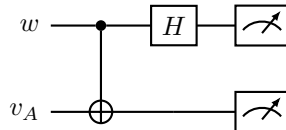
$$w_0 = |1\rangle, \quad w_1 = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \quad w_2 = \frac{1}{\sqrt{2}}(-|0\rangle + |1\rangle).$$

PART 2

In this part, justify your answer with appropriate amount of reasoning. Choose two out of the four problems from this part. You may choose more than two to answer, but we only use the best two ones among your answers to decide the score.

Problem 11. Recall the quantum teleportation scheme, as follows:

- (1) Alice and Bob share the state $v = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$: Alice keeps the first qubit factor v_A , and Bob keeps the second qubit factor v_B .
- (2) Given a qubit state w , Alice performs a unitary transform on $w \otimes v_A$ and the measurement as in the following quantum circuit diagram:



- (3) Alice sends the results of her measurement x and y to Bob.
- (4) Bob computes the unitary gate $U = U_y U_x$ according to

$$U_x = \begin{cases} I_2 & (x = 0) \\ X & (x = 1) \end{cases}, \quad U_y = \begin{cases} I_2 & (y = 0) \\ Z & (y = 1) \end{cases}$$

and transform his state by U .

Problem 11.a. What is Bob's state after Alice performs her measurement, but before he applies U ? In what sense is this scheme better than classical communication schemes?

Problem 11.b. If they share the state $u = |00\rangle$ at the beginning, can Bob use some choice of U according to x, y to achieve the same kind of communication? If so, give a concrete choice of U . If not, explain why it would not work.

Problem 11.c. Suppose that they share a state of the form $(\cos \epsilon)u + (\sin \epsilon)u'$ at the beginning, where u' is a state vector orthogonal to u . Then Alice starts with $w = |0\rangle$, then transforms her qubits using the same rule as usual. When Bob also follows the above protocol, then measures his qubit in the $|0\rangle, |1\rangle$ basis, what is the probability of him observing $|0\rangle$?

Problem 12. Recall the quantum money scheme: A "bill" has

- (1) a string of 0's and 1's: $i_1 \dots i_N$; and
- (2) a qubit state $v_1 \otimes \dots \otimes v_N$.

The "bank" knows a secret information: for each $1 \leq k \leq N$, whether one should measure v_k in the $|0\rangle, |1\rangle$ basis or the $|+\rangle, |-\rangle$ basis.

Problem 12.a. Suppose a bill has the string $i_1 i_2 = 01$ ($N = 2$), and a counterfeiter tries to guess the state $v_1 \otimes v_2$ by choosing either of the $|0\rangle, |1\rangle$ basis or the $|+\rangle, |-\rangle$ basis randomly for $j = 1, 2$, then measuring each factor v_j once. What are the possibilities of his guess, with which probability?

Problem 12.b. How does the bank authenticate bills? If a counterfeiter forges a bill as above, what is the probability of the bank being fooled by it? Given also an estimate for general N .

Problem 12.c. Can a counterfeiter perform measurement many times to improve his chance of guess?

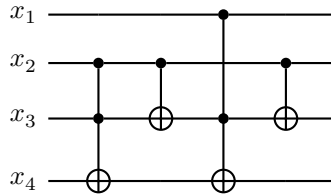
Problem 13. Given three bits x_1, x_2, x_3 , we define their *carry* to be

$$C(x_1, x_2, x_3) = \begin{cases} 1 & \text{at least two of the } x_i \text{ are 1} \\ 0 & \text{otherwise.} \end{cases}$$

A quantum gate implementing this, as

$$C' |x_1 x_2 x_3 x_4\rangle = |x_1 x_2 x_3 y\rangle \quad (y = \text{XOR}(x_4, C(x_1, x_2, x_3)))$$

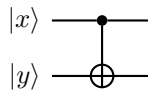
can be presented by the quantum circuit



Problem 13.a. Explain the correspondence between the above circuit and the formula

$$C(x_1, x_2, x_3) = \text{XOR}(\text{AND}(x_2, x_3), \text{AND}(x_1, \text{XOR}(x_2, x_3))).$$

Note that



can be interpreted as "store $\text{XOR}(x, y)$ in the second qubit", and we had

$$\text{XOR}(\text{XOR}(x, y), z) = \text{XOR}(x, \text{XOR}(y, z)).$$

Problem 13.b. Draw a quantum circuit with 4 qubits that can tell if there was a carry-over in the bit addition operation $x_1 + x_2 + x_3$.

Problem 14. Recall Grover's search problem: find the "marked input" $x^* \in \{0, 1\}^k$ specified by a Boolean function $f: \{0, 1\}^k \rightarrow \{0, 1\}$, where x^* is a solution for $f(x^*) = 1$. His algorithm was given by measurement of

$$v_j = (DV_f)^j |+\dots+\rangle$$

for an appropriate j , with the phase query operator

$$V_f |x\rangle = (-1)^{f(x)} |x\rangle$$

and the diffusion operator

$$D|x\rangle = \frac{1}{2^{k-1}} \sum_{y \in \{0,1\}^k} |y\rangle - |x\rangle.$$

Problem 14.a. Instead of $|+\cdots+\rangle$, suppose we used $v'_0 = \frac{1}{\sqrt{2}}(|y\rangle - |z\rangle)$ where $y, z \in \{0,1\}$ are not marked inputs. If we set $v'_j = (DV_f)^j v'_0$, can we find a marked input x^* with an appropriate choice of j ?

Problem 14.b. When there is an unknown number of marked inputs, we could fix a large T and randomly choose j between 1 and T . Since this choice is random, we might as well always choose $j = 1$ and do the measurement of v_1 . Do we get a good result this way?