UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT3440 — Dynamical Systems

Day of examination: Thursday 14 June 2018

Examination hours: 9:00 – 13:00

This problem set consists of 3 pages.

Appendices: None

Permitted aids: Approved calculator

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1 Linear systems (weight 15%)

Consider the linear system

$$\begin{cases} \dot{x} = Ax \\ x(0) = x_0. \end{cases} \tag{1}$$

For each of the following 2×2 matrices A, classify the equilibrium $x^* = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ of (1) and draw a phase portrait.

a

$$A = \begin{pmatrix} -1 & 1 \\ 2 & 0 \end{pmatrix}$$

b

$$A = \begin{pmatrix} 2 & 1 \\ -1 & 2 \end{pmatrix}$$

c

$$A = \begin{pmatrix} 2 & -1 \\ -1 & 2 \end{pmatrix}$$

Problem 2 Existence and uniqueness (weight 15%)

a

Find a solution of the scalar ODE

$$\begin{cases} \dot{x} = -x^2 \\ x(0) = x_0. \end{cases} \tag{2}$$

(Continued on page 2.)

Determine the maximal interval of existence of (2) for any $x_0 > 0$ — that is, the largest interval $(a, b) \subset \mathbb{R}$ so that a solution $x(t)$ of (2) exists for all $t \in (a, b)$.

b

Consider the scalar ODE

$$\begin{cases} \dot{x}(t) = \sin(x + t) \\ x(0) = x_0. \end{cases} \tag{3}$$

Are there more than one solution of (3)? Justify your answer, for example by applying one of the theorems from the course.

Problem 3 Numerical methods (weight 30%)

a Explicit and implicit Euler

Solve the ODE (2) numerically by computing a single step forward in time using both the explicit and implicit Euler methods. Use the parameters $h = \frac{1}{8}$ and $x_0 = \frac{9}{8}$.

b Stability of explicit and implicit Euler

Consider the linear system

$$\dot{x} = Ax, \qquad A = \begin{pmatrix} -101 & -100 \\ 1 & 0 \end{pmatrix}. \tag{4}$$

The matrix A has eigenvalues $\lambda_1 = -100$ and $\lambda_2 = -1$ (you don't have to show this). We approximate the linear system using both the explicit and implicit Euler methods. How will these two numerical methods behave if we use the step size $h = \frac{1}{10}$? What about $h = \frac{1}{100}$? Justify your answers.

c Truncation error

Consider the scalar, autonomous ODE

$$\dot{x} = f(x) \tag{5}$$

where $f : \mathbb{R} \to \mathbb{R}$ is a given function. Consider the Crank–Nicholson-type method

$$y_{n+1} = y_n + hf\Big(\frac{y_n + y_{n+1}}{2}\Big). \tag{6}$$

Estimate the truncation error $|\tau_{n+1}|$ of this method. You may assume that f is as smooth as you want.

Hint: If g and z are smooth functions then

$$\int_a^b g(z(s))\,ds = (b - a)g\Big(\frac{z(a) + z(b)}{2}\Big) + O((b - a)^3).$$

(Continued on page 3.)

Problem 4 Lotka–Volterra (weight 20%)

Consider the Lotka–Volterra model

$$\begin{cases} \dot{n} = n(2 - n - 2m) \\ \dot{m} = m(n - 1) \end{cases} \tag{7}$$

a

Which of the two unknowns represent the predator, and which the prey? Justify your answer.

b

The model (7) has an equilibrium at $(n_0^*, m_0^*) = \left(1, \frac{1}{2}\right)$. Find all the other equilibria.

c

Linearize (7) around the equilibrium (n_0^*, m_0^*). Use the linearized system to draw a phase portrait of the nonlinear system (7) around (n_0^*, m_0^*). Justify why the linearized system gives a good description of the behavior of (7) close to (n_0^*, m_0^*).

Problem 5 (weight 10%)

Consider the ODE

$$\begin{cases} \dot{u} = -v^3 \\ \dot{v} = u^3 \end{cases} \tag{8}$$

Find an expression for the orbits of (8) and draw a phase portrait.

Hint: The system (8) is Hamiltonian.

Problem 6 (weight 10%)

Show that the function $L(u, v) = u^2 + v^2$ is a Lyapunov function for the system

$$\begin{cases} \dot{u} = -v - uv^2 - u^3 \\ \dot{v} = u - v^3. \end{cases} \tag{9}$$

Use this to describe the long-time behavior of the solution (that is, explain what will happen to $x(t) = (u(t), v(t))$ when $t \to \infty$). Justify your answer.

THE END