UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in MAT3440 — Dynamical systems

Day of examination: Tuesday, June 11, 2019

Examination hours: 09:00 – 13:00

This problem set consists of 3 pages.

Appendices: None.

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1 (weight 30%)

Consider the differential equation

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} x - y^2 \\ x^2 - y \end{pmatrix}. \tag{1}$$

1a (weight 5%)

Find the two (there are only two) fixpoints for this system, and the linearization of (1) about these two fixpoints. What do the linearizations tell you about the stability of each fixpoint?

1b (weight 5%)

Show that the function

$$H(x, y) = \frac{1}{3}\left(x^3 + y^3\right) - xy$$

is an Hamiltonian for the system (1).

1c (weight 20%)

Draw a phase portrait of solutions to (1). The phase portrait must include both fixpoints and you should indicate the direction of the flow.

Problem 2 (weight 40%)

Consider the prey-predator model

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} x(2-x) - xy \\ -y + xy \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \tag{2}$$

where we assume that $x \geq 0$ and $y \geq 0$.

(Continued on page 2.)

2a (weight 5%)

Find all fixpoints of (2) and determine their type.

2b (weight 5%)

Find the solutions to (2) in the two cases

$$\textbf{i)}\ x_0 > 0,\ y_0 = 0, \text{ and } \textbf{ii)}\ x_0 = 0, y_0 > 0.$$

2c (weight 20%)

Let R denote the region

$$R = \{(x,y) \mid 0 \le x \le 3,\ 0 \le y \le 3 - x\}.$$

Prove that R is positively invariant for the flow defined by (2), i.e., that if $(x_0, y_0) \in R$, then $(x(t), y(t)) \in R$ for all $t > 0$.

2d (weight 10%)

If $y_0 = 3 - x_0$, and $x_0 \in (0,3)$, what can you say about the ω-limit set

$$\lim_{t\to\infty} \bigcup_{s>t} (x(s), y(s))\,?$$

Problem 3 (weight 30%)

Let f_α be given by

$$f_\alpha(x) = \alpha(1 - 2\,|x - 1/2|) = \begin{cases} 2\alpha x & x \le 1/2, \\ 2\alpha(1-x) & x \ge 1/2, \end{cases}$$

where α is a constant in the interval $(0,1)$. We also assume that $x \in [0,1]$. Consider the discrete dynamical system

$$x_{n+1} = f_\alpha(x_n), \quad n = 0, 1, 2, 3, \ldots, \tag{3}$$

with x_0 given.

3a (weight 5%)

Show that for $0 < \alpha < 1/2$, $x = 0$ is a stable fixpoint for the system (3).

3b (weight 5%)

Let $\alpha = 1/2$, find all fixpoints of (3) and determine whether they are stable or not.

3c (weight 5%)

Let $1/2 < \alpha < 1$, find all fixpoints for (3), and determine their stability.

(Continued on page 3.)

3d (weight 15%)

Let $1/2 < \alpha < 1$, show that there is a unique 2-periodic orbit for (3). Is this orbit stable? (Hint: Draw the graph of the second iteration of f, i.e. $f_\alpha^2 = f_\alpha \circ f_\alpha$, and find its local maxima and minima.)

THE END