UNIVERSITY OF OSLO

Faculty of Mathematics and Natural Sciences

Examination in MAT3440 — Dynamical systems

Day of examination: Tuesday, June 11, 2019

Examination hours: 09:00 – 13:00

This problem set consists of 6 pages.

Appendices: None.

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1 (weight 30%)

Consider the differential equation

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} x - y^2 \\ x^2 - y \end{pmatrix}. \tag{1}$$

1a (weight 5%)

Find the two (there are only two) fixpoints for this system, and the linearization of (1) about these two fixpoints. What do the linearizations tell you about the stability of each fixpoint?

Possible solution: The fixpoints are $(0,0)$ and $(1,1)$. We have that

$$DF(X) = \begin{pmatrix} 1 & -2y \\ 2x & -1 \end{pmatrix}.$$

Then we get

$$DF(0,0) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad DF(1,1) = \begin{pmatrix} 1 & -2 \\ 2 & -1 \end{pmatrix}.$$

The eigenvalues of 0 are $\lambda = \pm 1$, this is an unstable saddle for the linearization. Since the fixpoint is hyperbolic, this is an unstable saddle also for the nonlinear system. For the fixpoint at $(1,1)$ $\lambda = \pm i$, with zero real part. This tells you that for the nonlinear system, $(1,1)$ is either a stable or unstable fixpoint, or neutrally stable.

(Continued on page 2.)

1b (weight 5%)

Show that the function

$$H(x,y) = \frac{1}{3}\left(x^3 + y^3\right) - xy$$

is an Hamiltonian for the system (1).

Possible solution: We compute

$$\frac{\partial H}{\partial x} = x^2 - y, \text{ and } \frac{\partial H}{\partial y} = y^2 - x.$$

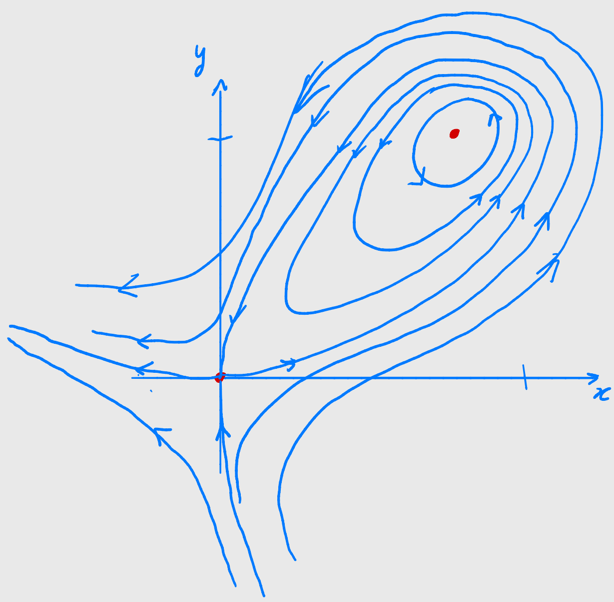
Then

$$\frac{d}{dt}H((x(t), y(t)) = \left(x^2 - y\right)\left(x - y^2\right) + \left(y^2 - x\right)\left(x^2 - y\right) = 0.$$

1c (weight 20%)

Draw a phase portrait of solutions to (1). The phase portrait must include both fixpoints and you should indicate the direction of the flow.

Possible solution: H has a local minimum at $(1,1)$, hence the solutions near $(1,1)$ are closed curves. From $DF(1,1)$ we see that this flow is counterclockwise. $(0,0)$ is a saddle point (no surprises here!). The portrait should look something like this:



(Continued on page 3.)

Problem 2 (weight 40%)

Consider the prey-predator model

$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} x(2-x) - xy \\ -y + xy \end{pmatrix}, \qquad \begin{pmatrix} x(0) \\ y(0) \end{pmatrix} = \begin{pmatrix} x_0 \\ y_0 \end{pmatrix}, \tag{2}$$

where we assume that $x \geq 0$ and $y \geq 0$.

2a (weight 5%)

Find all fixpoints of (2) and determine their type.

Possible solution: The fixpoints are $(0,0)$, $(2,0)$ and $(1,1)$. The Jacobian reads

$$DF = \begin{pmatrix} 2-y-2x & -x \\ y & x-1 \end{pmatrix},$$

so that

$$DF(0,0) = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}, \quad DF(2,0) = \begin{pmatrix} -2 & -2 \\ 0 & 1 \end{pmatrix}$$

$$\text{and} \quad DF(1,1) = \begin{pmatrix} -1 & -1 \\ 1 & 0 \end{pmatrix}.$$

The eigenvalues for $(0,0)$ are 2 and -1, hence this is an ustable saddle, the eigenvalues for $(2,0)$ are -2 and 1, hence this is an unstable saddle. The eigenvalues for $(1,1)$ are $(-1 \pm i\sqrt{3})/2$, so $(1,1)$ is a stable spiral, rotation counterclockwise (seen from $DF(1,1)_{1,2}$).

2b (weight 5%)

Find the solutions to (2) in the two cases

$$\textbf{i)}\ x_0 > 0,\ y_0 = 0, \text{ and } \textbf{ii)}\ x_0 = 0, y_0 > 0.$$

Possible solution: If $y_0 = 0$ then $y(t) = 0$ for all t. Hence x solves the logistic equation $x' = x(2-x)$, $x(0) = x_0$. This is separable, and we find its solution by integrating

$$\int_{x_0}^{x} \frac{dz}{z(2-z)} = \int_0^t ds,$$

with solution

$$x(t) = \frac{2x_0 e^t}{2 + x_0(e^t - 1)}.$$

If $x_0 = 0$, then $x(t) = 0$ for all t, and y solves $y' = -y$, so that $y(t) = y_0 e^{-t}$.

2c (weight 20%)

Let R denote the region

$$R = \{(x,y) \mid 0 \leq x \leq 3,\ 0 \leq y \leq 3 - x\}.$$

(Continued on page 4.)

Prove that R is positively invariant for the flow defined by (2), i.e., that if $(x_0, y_0) \in R$, then $(x(t), y(t)) \in R$ for all $t > 0$.

Possible solution: The boundary of R consists on three parts, one along the x-axis, one along the y-axis, and the line $y = 3 - x$ for $0 \le x \le 3$. By the previous question, the parts along the axis are orbits, so that solutions cannot leave R through this part of the boundary. Along the third part of the boundary orbits will enter R, this can be seen as follows: The outward pointing normal to this part of the boundary is $\mathbf{n} = (1, 1)$, solutions will enter R through this boundary if $\mathbf{n} \cdot F < 0$. We have that

$$\begin{aligned}\mathbf{n} \cdot F &= (1,1) \cdot (x(2-x) - xy, -y + xy) \\ &= x(2-x) - y = x(2-x) - (3-x) \\ &= -x^2 + 3x - 3 \\ &< 0, \quad \text{for all } x.\end{aligned}$$

2d (weight 10%)

If $y_0 = 3 - x_0$, and $x_0 \in (0, 3)$, what can you say about the ω-limit set

$$\lim_{t\to\infty} \bigcup_{s>t} (x(s), y(s))\,?$$

Possible solution: By the Poincaré-Bendixson theorem, this is either a limit cycle or the fixed point $(1, 1)$.

Problem 3 (weight 30%)

Let f_α be given by

$$f_\alpha(x) = \alpha(1 - 2\,|x - 1/2|) = \begin{cases} 2\alpha x & x \le 1/2, \\ 2\alpha(1-x) & x \ge 1/2, \end{cases}$$

where α is a constant in the interval $(0, 1)$. We also assume that $x \in [0, 1]$. Consider the discrete dynamical system

$$x_{n+1} = f_\alpha(x_n), \quad n = 0, 1, 2, 3, \ldots, \tag{3}$$

with x_0 given.

3a (weight 5%)

Show that for $0 < \alpha < 1/2$, $x = 0$ is a stable fixpoint for the system (3).

Possible solution: 0 is the only solution to $x = f_\alpha(x)$, $f'_\alpha(0) = 2\alpha < 1$ so this fixpoint is stable.

(Continued on page 5.)

3b (weight 5%)

Let $\alpha = 1/2$, find all fixpoints of (3) and determine whether they are stable or not.

Possible solution: Now

$$f_{1/2}(x) = \begin{cases} x & x \leq 1/2, \\ 1-x & x > 1/2, \end{cases}$$

so that the fixpoints are the interval $[0, 1/2]$. If we start near a fixpoint, we will not move further away (but not any closer either) so that the fixpoints are neutrally stable.

3c (weight 5%)

Let $1/2 < \alpha < 1$, find all fixpoints for (3), and determine their stability.

Possible solution: $x = 0$ is always a fixpoint, it is unstable since $f'_\alpha(0) = 2\alpha > 1$. Furthermore $f_\alpha(1/2) = \alpha > 1/2$ and $f_\alpha(1) = 0 < 1$. We also have that $f'_\alpha(x) < 0$ in $(1/2, 1)$ so that there is a unique solution x_α of $x = f_\alpha(x)$ in $(1/2, 1)$, $x_\alpha = 2\alpha/(1+2\alpha)$. Since $f'_\alpha(x_\alpha) = -2\alpha < -1$, this is an unstable fixpoint.

3d (weight 15%)

Let $1/2 < \alpha < 1$, show that there is a unique 2-periodic orbit for (3). Is this orbit stable? (Hint: Draw the graph of the second iteration of f, i.e. $f^2_\alpha = f_\alpha \circ f_\alpha$, and find its local maxima and minima.)

Possible solution: We have that $f_\alpha(1/2) = \alpha > 1/2$, so that f^2_α will have local maximi when $f_\alpha = 1/2$. solving the equation $f_\alpha(x) = 1/2$ we get

$$x = \begin{cases} 1/(4\alpha) = x_1 \\ 1 - 1/(4\alpha) = x_2. \end{cases}$$

f^2_α will have a local minimum when $x = 1/2$. So

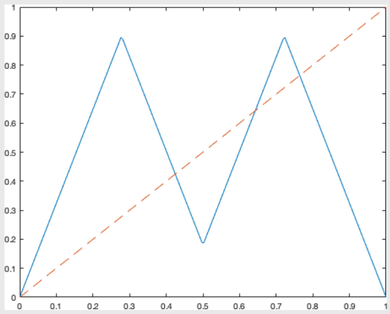
$$f^2_\alpha(1/2) = f_\alpha(\alpha) = 2\alpha(1-\alpha) < 1/2.$$

Now

$$\begin{aligned} f^2_\alpha(x_1) &= \alpha > \frac{1}{4\alpha}, \quad \text{since } \alpha > 1/2, \\ f^2_\alpha(1/2) &= 2\alpha(1-\alpha) < \frac{1}{2}, \\ f^2_\alpha(1 - 1/(4\alpha)) &= \alpha > 1 - \frac{1}{4\alpha}, \quad \text{since } \alpha > 1/2. \end{aligned}$$

(Continued on page 6.)

Since f_α^2 is continuous, the graph $y = f_\alpha^2(x)$ will intersect the graph $y = x$ once at z_1 in the interval $(x_1, 1/2)$, once at x_α (since x_α is a fixpoint of f_α it is a fixpoint of f_α^2) and once at z_2 in the interval $(x_2, 1)$. The 2-periodic orbit is $\{z_1, z_2\}$. Now f_α^2 is differentiable everywhere except for $x = x_1$, $x = 1/2$ and $x = x_2$, so $\left| f_\alpha^{2,\prime}(z_i) \right| = (2\alpha)^2 > 1$, $i = 1, 2$, therefore this orbit is unstable.



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