

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT3440 — Dynamical systems

Day of examination: Tuesday, June 6th, 2023

Examination hours: 09.00–13.00

This problem set consists of 3 pages.

Appendices: None

Permitted aids: No aids are allowed.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

a

Consider the following ordinary differential equation:

$$\frac{dx}{dt} = f_r(x) = rx + x^3,$$

where $r \in \mathbb{R}$ is a parameter. Compute all equilibria and determine their stability (source or sink). Draw a bifurcation diagram. What is the bifurcation value of r ?

b

Consider the differential equation

$$\frac{dx}{dt} = f(x) = \frac{1}{x},$$

where f is continuously differentiable (C^1) on $(0, \infty)$.

(i) Determine the flow $\phi_t(x_0)$ of this equation for $x_0 > 0$.

(ii) Determine the maximal interval of existence of the form $I(x_0) = (\alpha, \infty)$ for some α to be specified. Compute $\lim_{t \rightarrow \alpha+} \phi_t(x_0)$.

(iii) Verify that $\phi_{t+s}(x_0) = \phi_t(\phi_s(x_0))$ for $s, t \in I(x_0)$.

c

Construct two different solutions of the initial value problem

$$\frac{dx}{dt} = x^{1/3}, \quad x(0) = 0.$$

Why does the existence and uniqueness theorem fail to apply in this case?

(Continued on page 2.)

Problem 2**a**

Consider the following system of linear differential equations with parameter $b \neq 0$:

$$\frac{dx}{dt} = -x + by, \quad \frac{dy}{dt} = x - y.$$

Use *trace-determinant analysis* to determine the phase portrait at the origin $(0, 0)$.

b

Consider the following nonlinear system of differential equations:

$$\frac{dx}{dt} = -x - y^2, \quad \frac{dy}{dt} = x - y.$$

Determine the equilibrium solutions, and the phase portrait near each equilibrium solution.

c

Compute the matrix exponential of

$$A = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}.$$

Problem 3**a**

Consider the system

$$(1) \quad x' = y, \quad y' = -x - \varepsilon x^2 y,$$

where $\varepsilon > 0$ is a parameter. Explain why the equilibrium solution $(0, 0)$ cannot be analyzed using the linearization method.

b

(i) Formulate the Liapunov theorem for a general system $X' = F(X)$.

(ii) Apply the Liapunov theorem to determine the stability of the origin for the system (1).

c

(i) Formulate the the LaSalle invariance principle for a general system $X' = F(X)$.

(ii) Use the LaSalle invariance principle to show that $(0, 0)$ is an asymptotic stable equilibrium solution of the system (1).

(Continued on page 3.)

d

Consider the nonlinear system

$$x' = y, \quad y' = -\sin x.$$

- (i) Explain why this is a Hamiltonian system (you should specify the Hamiltonian so that $H(0,0) = 0$).
- (ii) Use the Liapunov theorem to verify the stability of the origin $(0,0)$. Explain why $(0,0)$ is not asymptotically stable.
- (iii) Establish the asymptotic stability of $(0,0)$ for the modified system

$$x' = y, \quad y' = -\sin x - \kappa y,$$

where κ is a strictly positive parameter.

THE END