

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT3440 — Dynamical systems

Day of examination: Tuesday, June 6th, 2023

Examination hours: 09.00–13.00

This problem set consists of 8 pages.

Appendices: None

Permitted aids: No aids are allowed.

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

a

Consider the following ordinary differential equation:

$$\frac{dx}{dt} = f_r(x) = rx + x^3,$$

where $r \in \mathbb{R}$ is a parameter. Compute all equilibria and determine their stability (source or sink). Draw a bifurcation diagram. What is the bifurcation value of r ?

Answer: The equilibrium solutions can be expressed as

$$x(r + x^2) = 0 \iff x = 0, \pm\sqrt{-r},$$

where the solution $x = 0$ exists for all values of r , while the solutions $x = \pm\sqrt{-r}$ exist only for $r < 0$. Considering the derivatives of f_r , we have $f'_r(0) = r$ and $f'_r(\pm\sqrt{-r}) = -2r$. Therefore, when $r < 0$, the equilibrium point $x = 0$ is stable (sink). Conversely, when $r > 0$, the equilibrium point $x = 0$ is unstable (source). Additionally, the equilibrium points $x = \pm\sqrt{-r}$ are unstable (source) for $r < 0$. A (subcritical pitchfork) bifurcation occurs at $r = 0$. The bifurcation diagram can be seen in Figure 1.

b

Consider the differential equation

$$\frac{dx}{dt} = f(x) = \frac{1}{x},$$

where f is continuously differentiable (C^1) on $(0, \infty)$.

(i) Determine the flow $\phi_t(x_0)$ of this equation for $x_0 > 0$.

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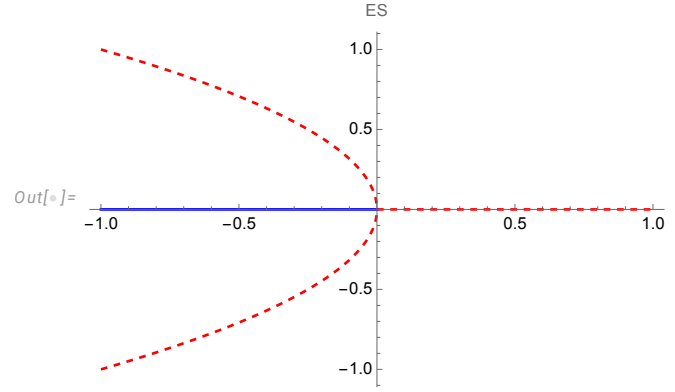


Figure 1: Problem 1a – bifurcation digram.

(ii) Determine the maximal interval of existence of the form $I(x_0) = (\alpha, \infty)$ for some α to be specified. Compute $\lim_{t \rightarrow \alpha+} \phi_t(x_0)$.

(iii) Verify that $\phi_{t+s}(x_0) = \phi_t(\phi_s(x_0))$ for $s, s+t \in I(x_0)$.

Answer: (i) It is easy to check that the solution $x(t)$ with initial condition $x(0) = x_0 > 0$ is given by

$$x(t) = \sqrt{2t + x_0^2}.$$

In other words, $\phi_t(x_0) = \sqrt{2t + x_0^2}$.

(ii) The maximal interval of existence is given by $2t + x_0^2 \geq 0$, so that

$$I(x_0) = \left(-\frac{x_0^2}{2}, \infty\right),$$

and, with $\alpha = -\frac{x_0^2}{2}$,

$$\lim_{t \rightarrow \alpha+} \phi_t(x_0) = 0.$$

(iii) Finally, directly from the solution formula, we have

$$\phi_{t+s}(x_0) = \sqrt{2t + 2s + x_0^2}$$

and

$$\begin{aligned} \phi_t(\phi_s(x_0)) &= \sqrt{2t + \phi_s(x_0)^2} = \sqrt{2t + \sqrt{2s + x_0^2}^2} \\ &= \sqrt{2t + 2s + x_0^2}, \end{aligned}$$

so that $\phi_{t+s}(x_0) = \phi_t(\phi_s(x_0))$.

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c

Construct two different solutions of the initial value problem

$$\frac{dx}{dt} = x^{1/3}, \quad x(0) = 0.$$

Why does the existence and uniqueness theorem fail to apply in this case?

Answer: Two possible solutions are

$$x(t) \equiv 0, \quad x(t) = \left(\frac{2}{3}\right)^{3/2} t^{3/2}.$$

The existence and uniqueness theorem does not apply to the initial value problem $x' = x^{1/3}$, $x(0) = 0$, because the function $f(x) = x^{1/3}$ is not continuously differentiable (or Lipschitz) at $x = 0$. This is due to the singularity of the derivative of $f(x)$ at $x = 0$.

Problem 2**a**Consider the following system of linear differential equations with parameter $b \neq 0$:

$$\frac{dx}{dt} = -x + by, \quad \frac{dy}{dt} = x - y.$$

Use *trace-determinant analysis* to determine the phase portrait at the origin $(0, 0)$.Answer: This system can be written as $X' = AX$ with $X = \begin{pmatrix} x \\ y \end{pmatrix}$ and the coefficient matrix

$$A = \begin{pmatrix} -1 & b \\ 1 & -1 \end{pmatrix}.$$

The trace T and determinant D of A are

$$T = -1 + (-1) = -2, \quad D = (-1)(-1) - (b)(1) = 1 - b.$$

Moreover,

$$T^2 - 4D = 4 - 4(1 - b) = 4b.$$

If $b > 0$, then $T^2 - 4D > 0$, which means that we have real eigenvalues. In this case, the origin will be a saddle point if $D < 0$, which is true when $b > 1$. If $D > 0$ and $T < 0$, which is true when $b < 1$, the origin will be a sink (stable).

If $b < 0$, then $T^2 - 4D < 0$, which means we have complex eigenvalues. However, since $T < 0$, the origin is a spiral sink.

(Continued on page 4.)

b

Consider the following nonlinear system of differential equations:

$$\frac{dx}{dt} = -x - y^2, \quad \frac{dy}{dt} = x - y.$$

Determine the equilibrium solutions, and the phase portrait near each equilibrium solution.

Answer: The equilibrium solutions are

$$(0, 0), \quad (-1, -1).$$

We analyze the stability of these equilibrium solutions using the linearization method. The Jacobian matrix of this system is

$$A(x, y) = \begin{pmatrix} -1 & -2y \\ 1 & -1 \end{pmatrix}$$

We now evaluate the Jacobian matrix at each equilibrium solution.

1. At the equilibrium solution $(0, 0)$:

$$A(0, 0) = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}.$$

The repeated eigenvalues of this matrix are $\lambda = -1$, which are real and negative. This means that the fixed point $(0, 0)$ is a stable sink.

2. At the equilibrium solution $(-1, -1)$:

$$A(-1, -1) = \begin{pmatrix} -1 & 2 \\ 1 & -1 \end{pmatrix}$$

The eigenvalues are $\lambda = -1 - \sqrt{2} < 0$ and $\lambda = -1 + \sqrt{2} > 0$. Hence, the point $(-1, -1)$ is a saddle.

c

Compute the matrix exponential of

$$A = \begin{pmatrix} -1 & 0 \\ 1 & -1 \end{pmatrix}.$$

Answer: The matrix exponential of A is defined by

$$\exp(A) = \sum_{k=0}^{\infty} \frac{A^k}{k!}.$$

One can check (via matrix multiplications) that

$$A^k = \begin{pmatrix} (-1)^k & 0 \\ k(-1)^{k+1} & (-1)^k \end{pmatrix}.$$

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We have

$$\sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = e^{-1}$$

and, noting that $k/k! = 1/(k-1)!$ and $(-1)^{k+1} = (-1)^{k-1}(-1)^2 = (-1)^{k-1}$ for $k = 1, 2, \dots$,

$$\sum_{k=0}^{\infty} \frac{k(-1)^{k+1}}{k!} = \sum_{k=1}^{\infty} \frac{(-1)^{k-1}}{(k-1)!} = \sum_{k=0}^{\infty} \frac{(-1)^k}{k!} = e^{-1}$$

Hence,

$$\exp(A) = \begin{pmatrix} e^{-1} & 0 \\ e^{-1} & e^{-1} \end{pmatrix}.$$

Problem 3

a

Consider the system

$$x' = y, \quad y' = -x - \varepsilon x^2 y, \quad (1)$$

where $\varepsilon > 0$ is a parameter. Explain why the equilibrium solution $(0, 0)$ cannot be analyzed using the linearization method.

Answer: Let us first recall that the linearization method is based on approximating the behavior of a nonlinear system near an equilibrium point using its linearization. The linearization is here obtained by computing the Jacobian matrix A of $F = (y, -x - \varepsilon x^2 y)$ evaluated at $(0, 0)$, and studying the eigenvalues of this matrix:

$$A = DF(0, 0) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix},$$

with eigenvalues $\lambda = \pm i$. Consequently, the system is not hyperbolic, and therefore the linearization method cannot determine the stability of the equilibrium point $(0, 0)$.

b

- (i) Formulate the Liapunov theorem for a general system $X' = F(X)$.
- (ii) Apply the Liapunov theorem to determine the stability of the origin for the system (1).

Answer: (i) The Lyapunov theorem takes the following form for a system

$$X'(t) = F(X(t))$$

of differential equations, where $F : \mathbb{R}^n \rightarrow \mathbb{R}^n$ is given. Let X^* be an equilibrium solution. Consider a function $L : O \rightarrow \mathbb{R}$ defined in some neighborhood O of X^* such that:

1. $L(X^*) = 0$,

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2. $L(X) > 0$ for all $X \in O$ such that $X \neq X^*$,
3. $DL(X) \cdot DF(X) \leq 0$ for all $X \in O$, $X \neq X^*$.

Then X^* is stable. Moreover, if

$$DL(X) \cdot DF(X) < 0$$

for all $X \in O$ such that $X \neq X^*$, then X^* is asymptotically stable.

(ii) To determine the stability of the origin using Liapunov's method, we use a Lyapunov function $L(x, y)$ given by

$$L(x, y) = \frac{1}{2} (x^2 + y^2).$$

Note that this Lyapunov function satisfies the two required conditions:

$$L(0, 0) = 0$$

and

$$L(x, y) > 0 \text{ for all } (x, y) \neq (0, 0).$$

Next, with $F = (y, -x - \varepsilon x^2 y)$, we compute

$$\begin{aligned} DL(x, y) \cdot DF(x, y) &= (x, y) \cdot (y, -x - \varepsilon x^2 y) \\ &= xy - yx - \varepsilon x^2 y^2 \\ &= -\varepsilon x^2 y^2 \leq 0. \end{aligned}$$

It follows from the Liapunov theorem that $(0, 0)$ is stable.

c

- (i) Formulate the the LaSalle invariance principle for a general system $X' = F(X)$.
- (ii) Use the LaSalle invariance principle to show that $(0, 0)$ is an asymptotic stable equilibrium solution of the system (1).

Answer: (i) Consider an equilibrium solution X^* of the system $X' = F(X)$. Let $L : O \rightarrow \mathbb{R}$ be a Liapunov function with $X^* \in O \subset \mathbb{R}^n$. Suppose $\mathcal{P} \subset O$ is a closed bounded positively invariant set. Consider the set

$$\mathcal{S} = \{X \in \mathcal{P} : DL(X) \cdot F(X) = 0\}.$$

Suppose that there exists no entire solution $\{X(t)\}_{t \in \mathbb{R}}$ that belongs to $\mathcal{S}$, except X^* . Then X^* is asymptotically stable.

- (ii) For the system (1), we may specify $O = \mathbb{R}^2$ and

$$\mathcal{P} = \{(x, y) \in \mathbb{R}^2 : x^2 + y^2 \leq 2r^2\} = \{(x, y) : L(x, y) \leq r\},$$

for some $r > 0$. Then clearly $(0, 0) \in \mathcal{P}$. Moreover, $\mathcal{P}$ is a positively invariant set since any solution $(x(t), y(t))$ satisfies

$$\frac{d}{dt} L(x(t), y(t)) = DL(x(t), y(t)) \cdot DF(x(t), y(t)) = -\varepsilon x^2 y^2 \leq 0;$$

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thus, $(x_0, y_0) \in \mathcal{P} \implies (x(t), y(t)) \in \mathcal{P}$ for all $t \geq 0$.

The set $\mathcal{S}$ takes the form

$$\mathcal{S} = \{(x, y) : -\varepsilon x^2 y^2 = 0\} = \{(x, y) : x = 0 \text{ or } y = 0\}.$$

Given the system (1), $x = 0$ implies that $y = 0$. Similarly, $y = 0$ implies that $x = 0$. In other words, there exists no entire solution $\{(x(t), y(t))\}_{t \in \mathbb{R}}$ that belongs to $\mathcal{S}$, except $(0, 0)$. Hence, by the LaSalle invariance principle, the origin $(0, 0)$ is asymptotic stable.

d

Consider the nonlinear system

$$x' = y, \quad y' = -\sin x.$$

(i) Explain why this is a Hamiltonian system (you should specify the Hamiltonian so that $H(0, 0) = 0$).

(ii) Use the Liapunov theorem to verify the stability of the origin $(0, 0)$. Explain why $(0, 0)$ is not asymptotically stable.

(iii) Establish the asymptotic stability of $(0, 0)$ for the modified system

$$x' = y, \quad y' = -\sin x - \kappa y,$$

where κ is a strictly positive parameter.

Answer: (i) The system is Hamiltonian since it takes the form

$$x' = H_y(x, y), \quad y' = -H_x(x, y),$$

where

$$H(x, y) = \frac{1}{2}y^2 + (1 - \cos x).$$

(ii) Clearly, the origin is an equilibrium solution. Observe that

$$H(0, 0) = 0, \quad H(x, y) > 0 \text{ for } (x, y) \neq (0, 0),$$

so that H is a Liapunov function. Moreover, with $F = (y, -\sin x)$,

$$DH(x, y) \cdot F(x, y) = (\sin x)y + y(-\sin x) = 0.$$

Hence, by the Liapunov theorem, the origin is stable. Since

$$\frac{d}{dt}H(x(t), y(t)) = DH(x(t), y(t)) \cdot F(x(t), y(t)) = 0,$$

the Hamiltonian is constant along the solution:

$$H(x(t), y(t)) = \text{constant for all } t \geq 0.$$

Consequently, $(0, 0)$ cannot be asymptotic stable.

(iii) Although this is not a Hamiltonian system, the function

$$H(x, y) = \frac{1}{2}y^2 + (1 - \cos x)$$

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can still be used as a Liapunov function. Set $F = (y, -\sin x - \kappa y)$. Then

$$DH(x, y) \cdot F(x, y) = -\kappa y^2 \leq 0,$$

so that the Liapunov theorem tells us that $(0, 0)$ is stable. To verify asymptotic stability, we use the LaSalle invariance principle, where a positively invariant set $\mathcal{P}$ can be specified as $\{(x, y) : H(x, y) \leq r\}$ for some $r > 0$. Note that $\mathcal{S}$ is the set of (x, y) for which $y = 0$. But if $y = 0$, then the modified system implies that also $\sin x = 0$, which implies that $x = 0$ whenever we are in a neighborhood of the origin. In other words, there exists no entire solution $\{(x(t), y(t))\}_{t \in \mathbb{R}}$ that belongs to $\mathcal{S}$, except $(0, 0)$. Thus, by the LaSalle invariance principle, the origin is asymptotically stable.

THE END