

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Examination in: MAT3440 — Dynamical systems

Day of examination: Friday, August 16, 2024

Examination hours: 15:00 – 19:00

This problem set consists of 4 pages.

Appendices: None.

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Problem 1

Consider the scalar differential equation

$$x'(t) + x(t) = \sin(t). \quad (1)$$

1a

Let $x = x(t; x_0)$ denote the solution which satisfies $x(0; x_0) = x_0$, also called the flow and denoted $\phi_t(x_0)$. Find a formula for the flow.

Possible solution: We have that

$$\phi_t(x_0) = \left(x_0 + \frac{1}{2}\right)e^{-t} + \frac{1}{2}(\sin(t) - \cos(t)).$$

1b

Let the sequence $\{x_n\}_{n \geq 0}$ be defined by

$$x_{n+1} = \phi_{2\pi}(x_n) \quad \text{for } n > 0,$$

where x_0 is given. Show that this sequence converges and find its limit.

Possible solution: This map is a contraction,

$$x \mapsto \phi_{2\pi}(x) = \left(x + \frac{1}{2}\right)e^{-2\pi} - \frac{1}{2}.$$

The limit is $-1/2$.

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1c

Explain why (1) has a unique 2π -periodic solution, and find this solution.

Possible solution: The unique periodic solution is $(\sin(t) - \cos(t))/2$.

Problem 2

Consider the dynamical system

$$\begin{aligned}x' &= x - y - x^3 \\y' &= x + y - y^3.\end{aligned}\tag{2}$$

2a

Find all fixpoints of (2) and determine their type.

Possible solution: The unique fixpoint is $(0, 0)$, and the eigenvalues of the linearized system are $1 \pm i$. Therefore it is a spiral source, counterclockwise rotation.

2b

Let the polar coordinates (r, θ) be defined by $x = r \cos(\theta)$ and $y = r \sin(\theta)$. Show that

$$\frac{1}{2} \frac{d}{dt} r^2 = r^2 (1 - r^2) + \frac{1}{2} r^2 \sin^2(2\theta).$$

2c

Show that for all $1 > \varepsilon > 0$

$$\left. \frac{d}{dt} r^2 \right|_{r=\sqrt{1-\varepsilon}} > 0, \quad \text{and} \quad \left. \frac{d}{dt} r^2 \right|_{r=\sqrt{2+\varepsilon}} < 0.$$

Possible solution: If $r^2 = 1 - \varepsilon$

$$\frac{d}{dt} r^2 = (1 - \varepsilon) \left(\varepsilon + \frac{1}{2} \sin^2(2\theta) \right) \geq (1 - \varepsilon) \varepsilon > 0.$$

If $r^2 = 2 + \varepsilon$

$$\frac{d}{dt} r^2 = -(2 + \varepsilon)(1 + \varepsilon) - (2 + \varepsilon)(1 - \frac{1}{2} \sin^2(2\theta)) \leq -(2 + \varepsilon)(1 + \varepsilon) < 0.$$

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2d

Explain why this implies that the annulus $\{r \mid 1 \leq r \leq \sqrt{2}\}$ contains a periodic orbit.

Possible solution: The annulus $\{r \mid 1 - \varepsilon < r < \sqrt{2} + \varepsilon\}$ is positively invariant for all $\varepsilon > 0$ and contains no fixpoints. Therefore it contains a periodic orbit. Let $\varepsilon \rightarrow 0$ to conclude.

Problem 3

Recall the numerical schemes for the differential equation $x' = f(x)$;

$$\begin{aligned} x_{n+1} &= x_n + hf(x_n) && \text{(explicit Euler),} \\ x_{n+1} &= x_n + hf(x_{n+1}) && \text{(implicit Euler).} \end{aligned}$$

Consider the initial value problem

$$\begin{cases} x' = x^{\frac{1}{2}} & t > 0, \\ x(0) = 0. \end{cases} \quad (3)$$

3a

Find all solutions of (3).

Possible solution: For any positive c ,

$$u_c(t) = \begin{cases} 0 & 0 \leq t \leq c, \\ \frac{1}{4}(t - c)^2 & t > c, \end{cases}$$

is a solution.

3b

Show that the explicit Euler method fails to approximate the solution of (3) which satisfies $x(1) = 1/4$.

Possible solution: We get $x_0 = 0$ and therefore $x_n = 0$ for all n .

3c

Let now $\{x_n\}_{n \geq 0}$ be generated by the implicit Euler method and make the ansatz $x_n = (C_n nh)^2$. Show that there is a solution (of the implicit Euler

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scheme) with $C_0 = 0$, $C_1 = 1$. Find a formula for C_n ,

$$C_{n+1} = F(n, C_n).$$

Extra question (difficult): Show that $C_{n+1} < C_n$ for all $n \geq 1$.

Possible solution: We solve the implicit scheme using the above ansatz

$$C_{n+1} = \frac{1}{2(n+1)} \left(1 + \sqrt{1 + 4C_n^2 n^2} \right), \quad n \geq 0.$$

Let S_n solve $F(n, S_n) = S_n$, we get

$$S_n = \frac{n}{2n+1},$$

so that $S_n \nearrow 1/2$. We also have that

$$0 < \frac{\partial F}{\partial C} = \frac{n}{n+1} \frac{2Cn}{\sqrt{1 + (2Cn)^2}} < \frac{n}{n+1} < 1,$$

so that if $C_n > S_n$, $S_n < C_{n+1} < C_n$. If $C_n > 1/2$, then

$$\begin{aligned} C_{n+1} &= F(n, C_n) \\ &> F\left(n, \frac{1}{2}\right) = \frac{1}{2(n+1)} \left(1 + \sqrt{1 + n^2} \right) \\ &> \frac{1}{2(n+1)} (1 + n) = \frac{1}{2} \\ &> S_{n+1}. \end{aligned}$$

Therefore by induction $1/2 < C_{n+1} < C_n$ and $C_n \searrow C$ for some $C \geq 1/2$. Now

$$\begin{aligned} C - \frac{1}{2} &= \lim_n C_{n+1} - \frac{1}{2} = \lim_n \left(F(n, C_n) - \frac{1}{2} \right) \\ &\leq \lim_n |F(n, C_n) - F(n, C)| + \lim_n \left| F(n, C) - \frac{1}{2} \right| \\ &\leq \lim_n |C_n - C| + \lim_n \left| F(n, C) - \frac{1}{2} \right| \\ &= 0. \end{aligned}$$

THE END

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