

# UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT3440 — Dynamical Systems

Day of examination: Wednesday 28th May 2025

Examination hours: 15:00–19:00

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

**For each question you may use results from the preceding questions even if you haven't answered them. The questions are not necessarily ordered by increasing difficulty.**

## Problem 1 (weight 10%)

Recall that a function  $f : \mathbb{R} \rightarrow \mathbb{R}$  is said to be periodic of period  $T > 0$  if for all  $t \in \mathbb{R}$  we have:

$$f(t + T) = f(t). \quad (1)$$

Let  $T > 0$ . Consider an arbitrary continuous function  $p : \mathbb{R} \rightarrow \mathbb{R}$  which is periodic of period  $T$  and such that:

$$\int_0^T p(t) dt = 0. \quad (2)$$

We consider now the following ordinary differential equation for a function  $x : \mathbb{R} \rightarrow \mathbb{R}$ :

$$\text{for all } t \in \mathbb{R} \quad x'(t) = p(t)x(t). \quad (3)$$

Show that the function  $x$  is periodic with period  $T$ .

## Problem 2 (weight 10%)

For each  $\alpha \in \mathbb{R}$  we denote by  $A(\alpha)$  the following  $2 \times 2$  matrix:

$$A(\alpha) = \begin{bmatrix} 0 & -1 \\ 1 & \alpha \end{bmatrix}. \quad (4)$$

For each  $\alpha$  we have a differential equation:

$$X' = A(\alpha)X. \quad (5)$$

Discuss the nature of the phase portrait of this differential equation, depending on the value of  $\alpha$ . You may introduce the trace-determinant plane.

*(Continued on page 2.)*

**Problem 3** (weight 10%)

For any parameter  $a \in \mathbb{R}$  we consider the differential equation:

$$x'(t) = ax(t) - x(t)^5. \quad (6)$$

Verify that for a certain value of the parameter  $a$  the system undergoes a bifurcation. Analyse in particular the number, the location, and the nature of equilibrium points.

**Problem 4** (weight 20%)

Fix some real number  $a < 0$ . We consider the differential equation:

$$x' = ax. \quad (7)$$

**a** (weight 10%)

For a given time-step  $h > 0$ , write down explicitly what the numerical methods known as the "forward Euler" (also called explicit Euler), the "backward Euler" (also called implicit Euler), and "midpoint rule" (also called modified Euler) give in this case.

**b** (weight 10%)

Let  $h > 0$  be the size of the time step, as before. For each of these three numerical methods discuss how large  $h$  can be, as a function of  $a$ , while the numerical solution has the property of remaining bounded for all positive times, for any initial condition.

**Problem 5** (weight 20%)**a** (weight 10%)

Let  $V : \mathbb{R} \rightarrow \mathbb{R}$  be defined by:

$$V(x) = (x^2 - 1)^2. \quad (8)$$

We consider the differential equation:

$$x' = -V'(x). \quad (9)$$

Find all the equilibrium points of this equation. For each of them discuss whether they are unstable, stable and/or asymptotically stable.

**b** (weight 10%)

Let  $V : \mathbb{R}^2 \rightarrow \mathbb{R}$  be defined by:

$$V(x, y) = (x^2 + y^2 - 1)^2. \quad (10)$$

(Continued on page 3.)

We consider the system of differential equations:

$$\begin{cases} x' &= -\frac{\partial V}{\partial x}(x, y), \\ y' &= -\frac{\partial V}{\partial y}(x, y). \end{cases} \quad (11)$$

Find all the equilibrium points of this equation. For each equilibrium point discuss whether they are unstable, stable and/or asymptotically stable.

### Problem 6 (weight 30%)

We consider the following generalization of the Lotka Volterra model. We suppose we have two smooth functions  $M : \mathbb{R}^2 \rightarrow \mathbb{R}$  and  $L : \mathbb{R}^2 \rightarrow \mathbb{R}$ . We consider the system:

$$\begin{cases} x' &= -\frac{\partial L}{\partial y}(x, y)M(x, y), \\ y' &= \frac{\partial L}{\partial x}(x, y)M(x, y). \end{cases} \quad (12)$$

We suppose furthermore that if  $x = 0$  or  $y = 0$ , then  $M(x, y) = 0$ .

#### a (weight 10%)

Show that the function  $L$  is constant on any solution to the differential equation.

#### b (weight 10%)

Show that the region in the plane determined by  $x > 0$  and  $y > 0$  is invariant under the flow, forward in time.

#### c (weight 10%)

We suppose that  $L$  has a strict minimum in a point  $(x_0, y_0)$ , with  $x_0 > 0$  and  $y_0 > 0$ . Show that the differential equation has an equilibrium in  $(x_0, y_0)$ , that it is stable but not asymptotically stable.