

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT3440 — Dynamical Systems

Day of examination: Wednesday 28th May 2025

Examination hours: 15:00–19:00

This problem set consists of 7 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

For each question you may use results from the preceding questions even if you haven't answered them. The questions are not necessarily ordered by increasing difficulty.

Problem 1 (weight 10%)

Recall that a function $f : \mathbb{R} \rightarrow \mathbb{R}$ is said to be periodic of period $T > 0$ if for all $t \in \mathbb{R}$ we have:

$$f(t + T) = f(t). \quad (1)$$

Let $T > 0$. Consider an arbitrary continuous function $p : \mathbb{R} \rightarrow \mathbb{R}$ which is periodic of period T and such that:

$$\int_0^T p(t) dt = 0. \quad (2)$$

We consider now the following ordinary differential equation for a function $x : \mathbb{R} \rightarrow \mathbb{R}$:

$$\text{for all } t \in \mathbb{R} \quad x'(t) = p(t)x(t). \quad (3)$$

Show that the function x is periodic with period T .

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We have:

$$x(t) = x(0) \exp\left(\int_0^t p(s) ds\right). \quad (4)$$

Therefore we get:

$$x(t+T) = \exp\left(\int_t^{t+T} p(s) ds\right)x(t). \quad (5)$$

But we have:

$$\int_0^T p(s) ds = 0, \quad (6)$$

and:

$$\frac{d}{dt} \int_t^{t+T} p(s) ds = p(t+T) - p(t) = 0. \quad (7)$$

This gives:

$$\int_t^{t+T} p(s) ds = 0, \quad (8)$$

Since $\exp(0) = 1$ we are done.

Alternative intermediate step:

$$\int_0^{t+T} p(s) ds = \int_T^{t+T} p(s) ds, \quad (9)$$

$$= \int_T^{t+T} p(s-T) ds, \quad (10)$$

$$= \int_0^t p(s) ds. \quad (11)$$

Problem 2 (weight 10%)

For each $\alpha \in \mathbb{R}$ we denote by $A(\alpha)$ the following 2×2 matrix:

$$A(\alpha) = \begin{bmatrix} 0 & -1 \\ 1 & \alpha \end{bmatrix}. \quad (12)$$

For each α we have a differential equation:

$$X' = A(\alpha)X. \quad (13)$$

Discuss the nature of the phase portrait of this differential equation, depending on the value of α . You may introduce the trace-determinant plane.

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Denote by χ_α the characteristic polynomial of $A(\alpha)$. We have:

$$\chi_\alpha(\lambda) = \begin{vmatrix} \lambda & 1 \\ -1 & \lambda - \alpha \end{vmatrix} = \lambda^2 - \alpha\lambda + 1. \quad (14)$$

The discriminant is $\Delta = \alpha^2 - 4$. The (possibly complex) eigenvalues are given by:

$$(\alpha \pm \sqrt{\Delta})/2. \quad (15)$$

One can visualize this in the trace-determinant plane. The trace of $A(\alpha)$ is α and the determinant is 1. The points $(\alpha, 1)$ with $\alpha \in \mathbb{R}$ constitute a horizontal line through the point $(0, 1)$. It crosses the parabola of zero discriminant ($\Delta = 0$) at the values $\alpha = \pm 2$.

We get the following cases:

- $\alpha > 2$: source.
- $\alpha = 2$: source with linear term (only one eigenvector).
- $0 < \alpha < 2$: spiral source.
- $0 = \alpha$: center/orbits.
- $-2 < \alpha < 0$: spiral sink.
- $\alpha = -2$: sink with linear term.
- $\alpha < -2$: sink.

Problem 3 (weight 10%)

For any parameter $a \in \mathbb{R}$ we consider the differential equation:

$$x'(t) = ax(t) - x(t)^5. \quad (16)$$

Verify that for a certain value of the parameter a the system undergoes a bifurcation. Analyse in particular the number, the location, and the nature of equilibrium points.

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Equilibria are given by: $ax - x^5 = 0$, corresponding to $x = 0$ and $x^4 = a$.

If $a \leq 0$ there is only one equilibrium point, namely 0. When $a < 0$ linearization shows that it is asymptotically stable.

When $a = 0$ linearization does not allow one to conclude but we can compute the explicit solution:

$$x(t) = \text{sign}(x(0))(4t + |x(0)|)^{-1/4}. \quad (17)$$

This shows that the origin is asymptotically stable. One can also use the Lyapunov function $V(x) = x^2$.

If $a > 0$ the equilibria are 0 and the two points $\pm a^{1/4}$. Notice that the two complex roots are excluded. The origin is an unstable equilibrium. Linearization is given at x by $L_a(x) = a - 5x^4$. At both $\pm a^{1/4}$ it is $-4a$ which is negative giving an asymptotically stable equilibrium.

Problem 4 (weight 20%)

Fix some real number $a < 0$. We consider the differential equation:

$$x' = ax. \quad (18)$$

a (weight 10%)

For a given time-step $h > 0$, write down explicitly what the numerical methods known as the "forward Euler" (also called explicit Euler), the "backward Euler" (also called implicit Euler), and "midpoint rule" (also called modified Euler) give in this case.

[3] Forward Euler:

$$x_{n+1} - x_n = hax_n, \quad (19)$$

giving:

$$x_{n+1} = (1 + ha)x_n. \quad (20)$$

[3] Backward Euler:

$$x_{n+1} - x_n = hax_{n+1}, \quad (21)$$

giving:

$$x_{n+1} = (1 - ha)^{-1}x_n. \quad (22)$$

[4] Midpoint Euler:

$$x_{n+1} - x_n = ha(x_{n+1} + x_n)/2, \quad (23)$$

giving:

$$x_{n+1} = (1 + ha/2)(1 - ha/2)^{-1}x_n. \quad (24)$$

Note: several definitions of "modified Euler" were acceptable, as long as the reasoning was consistent.

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b (weight 10%)

Let $h > 0$ be the size of the time step, as before. For each of these three numerical methods discuss how large h can be, as a function of a , while the numerical solution has the property of remaining bounded for all positive times, for any initial condition.

- [3] Forward Euler: Requirement is $|1 + ha| \leq 1$. Gives $h \leq -2/a$.
 [3] Backward Euler: Requirement is $|1 - ha|^{-1} \leq 1$. Since $1 - ha \geq 1$ it is fulfilled for all $h \geq 0$. The method is unconditionally stable.
 [4] Midpoint Euler: Requirement is $|(1 + ha/2)(1 - ha/2)^{-1}| \leq 1$. But we have both $1 + ha/2 \leq 1 - ha/2$ and $-1 - ha/2 \leq 1 - ha/2$, so this is also always fulfilled.

Problem 5 (weight 20%)

a (weight 10%)

Let $V : \mathbb{R} \rightarrow \mathbb{R}$ be defined by:

$$V(x) = (x^2 - 1)^2. \quad (25)$$

We consider the differential equation:

$$x' = -V'(x). \quad (26)$$

Find all the equilibrium points of this equation. For each of them discuss whether they are unstable, stable and/or asymptotically stable.

Equilibria are -1 (asymptotically stable), 0 (unstable), and 1 (asymptotically stable).

b (weight 10%)

Let $V : \mathbb{R}^2 \rightarrow \mathbb{R}$ be defined by:

$$V(x, y) = (x^2 + y^2 - 1)^2. \quad (27)$$

We consider the system of differential equations:

$$\begin{cases} x' &= -\frac{\partial V}{\partial x}(x, y), \\ y' &= -\frac{\partial V}{\partial y}(x, y). \end{cases} \quad (28)$$

Find all the equilibrium points of this equation. For each equilibrium point discuss whether they are unstable, stable and/or asymptotically stable.

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Remark: The potential V is often referred to as the Mexican hat. It forces solutions to converge towards a region near the origin, but not to the origin itself...

There is an unstable equilibrium at 0. The other equilibria constitute the unit circle. Going to polar coordinates we obtain a one-dimensional system similar to the previous. Points on the unit circle are stable but not asymptotically stable (since all points on the circle are fixed by the flow).

Problem 6 (weight 30%)

We consider the following generalization of the Lotka Volterra model. We suppose we have two smooth functions $M : \mathbb{R}^2 \rightarrow \mathbb{R}$ and $L : \mathbb{R}^2 \rightarrow \mathbb{R}$. We consider the system:

$$\begin{cases} x' &= -\frac{\partial L}{\partial y}(x, y)M(x, y), \\ y' &= \frac{\partial L}{\partial x}(x, y)M(x, y). \end{cases} \quad (29)$$

We suppose furthermore that if $x = 0$ or $y = 0$, then $M(x, y) = 0$.

a (weight 10%)

Show that the function L is constant on any solution to the differential equation.

We compute:

$$\frac{d}{dt}L(x(t), y(t)) = \frac{\partial L}{\partial x}x'(t) + \frac{\partial L}{\partial y}y'(t), \quad (30)$$

$$= -\frac{\partial L}{\partial x} \frac{\partial L}{\partial y}(x, y)M(x, y) + \frac{\partial L}{\partial y} \frac{\partial L}{\partial x}(x, y)M(x, y), \quad (31)$$

$$= 0. \quad (32)$$

b (weight 10%)

Show that the region in the plane determined by $x > 0$ and $y > 0$ is invariant under the flow, forward in time.

Any solution that starts in this region and somehow exits it, would have to cross one of the coordinate axes. By M is zero on both axes, so they consist of equilibria. By uniqueness any solution that has a point on the axes would need to be constant in time, so it cannot start in the specified region.

c (weight 10%)

We suppose that L has a strict minimum in a point (x_0, y_0) , with $x_0 > 0$ and $y_0 > 0$. Show that the differential equation has an equilibrium in (x_0, y_0) ,

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that it is stable but not asymptotically stable.

The function L is a Lyapunov function. This shows that the point is a stable equilibrium.

It is not asymptotically stable because if $(x(t), y(t)) \rightarrow (x_0, y_0)$ then $L(x(t), y(t)) \rightarrow L(x_0, y_0)$. But the left hand side is constant giving $L(x(0), y(0)) = L(x_0, y_0)$. Since (x_0, y_0) is a strict minimum this would imply $(x(0), y(0)) = (x_0, y_0)$.