Oppgaver MAT 3440 14/3.

7.1

d) Explicit solution:

$$\frac{x'}{\cos(x)} = 1 \qquad \int_c^x \frac{\cos(x)}{1-\sin^2 x}\,dx = t \quad \leftarrow \text{separation.}$$

$$u = \sin x \qquad + \int \frac{du}{1-u^2} = \frac{1}{2}\ln\left(\frac{1-u}{1+u}\right) + c.$$

$$\Rightarrow \quad \frac{1-u}{1+u} = c^2 e^{2t} \qquad u = \frac{1-c^2e^{2t}}{1+c^2e^{2t}} \qquad u(0)=0 \Rightarrow c=1.$$

back to $\sin(x)$

$$\sin(x) = \frac{-\sinh(t)}{\cosh(t)} \qquad x = \sin^{-1}(-\tanh(t))$$

Picard iteration.

$$x_0 = 0$$

$$x_1 = 0 + \int_0^t \cos(0)\,dt = t$$

$$x_2 = \int_0^t \cos(t)\,dt = \sin(t)$$

$$x_3 = \int_0^t \cos(\sin(t))\,dt = \dots$$

e) Explicit solution

$$x' = \frac{1}{2x} \qquad 2xx' = \frac{d}{dt}(x^2) = 1$$

$$x(t)^2 - x(1)^2 = \int_1^t s\,ds = t-1$$

$$x(t) = \sqrt{t}$$

Picard iteration

$$x_0 = 1$$

$$x_1 = 1 + \int_1^t \frac{1}{2}\,ds = 1 + \frac{t}{2} - \frac{1}{2} = \frac{1}{2}(t+1)$$

$$x_2 = 1 + \int_1^t \frac{ds}{s+1} = 1 + \ln(t+1) - \ln(2)$$

$$x_3 = 1 + \int_0^t \frac{1}{2(1-\ln(2)+\ln(s+1))}\,ds = \dots$$

7.2

$$X_0 = X_0 = A^0 X_0.$$

$$X_1 = X_0 + \int_0^t A X_0 \, ds = X_0 + A X_0 t$$

$$X_2 = X_0 + \int_0^t A X_0 + A^2 X_0 s \, ds = X_0 + A X_0 + \frac{1}{2} t^2 A^2 X_0$$

$$X_3 = \vdots \ldots$$

Assume that

$$X_k = \sum_{j=0}^{k} \frac{t^j}{j!} A^k X_0$$

Then it is easy to show (by induction) that

$$X_{k+1} = \sum_{j=0}^{k+1} \frac{t^j}{j!} A^j X_0 .$$

7.6

$$x' = x^a \qquad x(0) = 0$$

$x(t) = 0$ is a solution for all $a > 0$.

$a > 1$ If $x(t) \neq 0$ for some t, then by continuity we have that $x(t) > 0$ for $t \in (t_0, t_1)$ where $x(t_0) = 0$.

$$\int_{t_0}^{t} \frac{x'}{x^a} \, ds = \frac{1}{1-a} x^{1-a} = t - t$$

but $-\infty = \lim_{t \searrow t_0} \frac{1}{1-a} x^{1-a}(t)$ and $0 = \lim_{t \searrow t_0} (t - t_0)$

so $x(t) = 0$ is the only solution.

If $a<1$ then

$\frac{1}{1-a} x^{1-a}(t) = t$ is a solution.

or $x(t) = \left(t(1-a)\right)^{\frac{1}{1-a}}$.

<u>But</u>

$$x_c(t) = \begin{cases} 0 & t \le c \\ \left((t-c)(1-a)\right)^{\frac{1}{1-a}} & t > c \end{cases}$$

is a solution for all $c > 0$.

<u>Moral</u>

$a \geq 1$ one solution $x = 0$

$a < 0$ many solutions.