

MANDATORY ASSIGNMENT

MAT3440—SPRING 2023

INFORMATION

All mandatory assignments must be uploaded via Canvas.

- *The assignment must be submitted as a single PDF file.*
- *Scanned pages must be clearly readable.*
- *The submission must contain your name and course code.*

If these requirements are not met, the assignment will not be evaluated. Read the information about mandatory assignments carefully: <http://www.uio.no/english/studies/examinations/compulsory-activities/mn-math-mandatory.html>

*In order to be eligible for the final examination, it is necessary to pass this assignment, which allows only **one** attempt to pass. It is particularly crucial that you demonstrate a clear attempt to solve all of the problems within the assignment. Provide thorough calculations and reasoning to support each answer.*

PROBLEM 1

a)

(i) Solve the differential equation

$$y'(t) = y(t)^2 + 1, \text{ for } t > 0,$$

with initial condition $y(0) = 1$. What is the largest interval $[0, T)$ on which this solution exists?

(ii) According to The Existence and Uniqueness Theorem from the book (page 387), there exists a unique solution of the above differential equation for $t \in [0, a)$ for some (possibly small) $a > 0$. Explain why a cannot be smaller than the T you found.

b)

Another way of representing solutions of the differential equation $x' = f(x)$ is via the so-called flow map $\phi_t(x_0) = \phi(t, x_0)$. In this representation, $\phi_t(x_0)$ is the solution $x(t)$ at time t starting from the initial condition x_0 . Therefore, the flow map provides a complete picture of the time evolution of *all* solutions to the differential equation! For autonomous equations $x' = f(x)$, where f has no explicit dependence on time t , the solution $x(t)$ satisfies the so-called semi-group property:

$$\phi_{t+s}(x_0) = \phi_t(\phi_s(x_0)) = \phi_s(\phi_t(x_0)).$$

The semi-group property states that the flow has a group-like structure, where the composition of two flows ϕ_t and ϕ_s is equivalent to the flow ϕ_{t+s} . Intuitively, the time evolution of the solution does not depend on how we split the time interval $[0, t + s]$ into two parts $[0, t]$ and $[t, t + s]$.

Verify the semi-group property by using the uniqueness of solutions to the differential equation $x' = f(x)$.

c)

Find and classify the equilibrium solutions of the following differential equation:

$$x' = f(x) := (x^2 - 4)(x^2 + 2x + 1).$$

Plot (on a computer) the slope field.

d)

Consider the differential equations

(i) $x' = f_c(x) := x^2 + (c - 1)x,$

(ii) $x' = g_c(x) := x^3 - x^2 + cx,$

where c is a parameter. Find and classify the equilibrium solutions. Sketch bifurcation diagrams using dashed lines for unstable equilibria and solid lines for stable equilibria. Add appropriate vertical lines with arrows to show whether the solutions are moving towards or away from the equilibria. What are the bifurcation values?

e)

Consider the non-autonomous linear differential equation

$$(1) \quad x' = -x + \sin t.$$

The equation exhibits periodicity in t , with a period $T = 2\pi$. It can be interpreted as a model for a quantity $x(t)$ that undergoes exponential decay (represented by the " $-x$ " term) and experiences periodic replenishment or depletion at a rate of $\sin t$.

Determine the Poincaré map $P(x_0)$, using the Variation of Parameters formula. Determine the fixed point of P and the solution of (1) with this fixed point as initial condition. What happens with any other solution as $t \rightarrow \infty$? Plot a range of solutions for various initial values in $[-1, 1]$, including the fixed point of the Poincaré map, and $t \in [0, 7]$.

PROBLEM 2

a)

Use trace—determinant analysis to decide if the linear system

$x' = Ax$ has a saddle, (spiral) sink, (spiral) source or center at the origin:

$$(i) \quad A = \begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}, \quad (ii) \quad A = \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix},$$

$$(iii) \quad A = \begin{pmatrix} 0 & -1 \\ 2 & -3 \end{pmatrix}, \quad (iv) \quad A = \begin{pmatrix} \gamma & \gamma \\ -\gamma & \gamma \end{pmatrix}, \quad \gamma \neq 0.$$

b)

Let A, B be two matrices. State a condition under which we have that

$$(1) \quad \exp(A + B) = \exp(A)\exp(B) = \exp(B)\exp(A).$$

Use (1) to compute $\exp \begin{pmatrix} a & b \\ -b & a \end{pmatrix}$.

c)

Use the matrix exponential and (1) to solve the system

$$x' = -rx + y, \quad y' = -ry, \quad x(0) = x_0, y(0) = y_0,$$

where $r > 0$. Plot the phase portrait for $r = 1$.

d)

Consider the problem

$$x' = -rx - y, \quad y' = x - ry, \quad x(0) = x_0, y(0) = y_0,$$

where r is a parameter.

(i) Use the matrix exponential method to show that

$$(2) \quad x^2(t) + y^2(t) = e^{-2rt} (x_0^2 + y_0^2).$$

Consequently,

$$x(t), y(t) \rightarrow 0 \text{ as } t \rightarrow \infty \text{ if } r > 0,$$

and

$$|x(t)|, |y(t)| \rightarrow \infty \text{ as } t \rightarrow \infty \text{ if } r < 0.$$

(ii) Use the differential equations and the chain rule for derivatives to prove (2) directly.

PROBLEM 3

We are interested in the explicit Euler method for solving the differential equation

$$(1) \quad x'(t) = f(x(t)), \quad t \in (0, T], \quad x(0) = x_0,$$

where $T > 0$ is an arbitrary fixed final time.

Going forward, let us make the assumption that $f : [0, 1] \rightarrow \mathbb{R}$ is a continuous function satisfying the so-called *dissipative condition*

$$(2) \quad f(0) = 0, \quad -K(x_2 - x_1) \leq f(x_2) - f(x_1) \leq 0, \quad \forall x_1, x_2 \in [0, 1], x_1 \leq x_2,$$

where $K > 0$ is a constant. We assume that the initial value x_0 always lies within the unit interval $[0, 1]$:

$$(3) \quad 0 \leq x_0 \leq 1.$$

The Euler method computes an approximate solution by using the following iteration:

$$y_{n+1} = y_n + hf(y_n), \quad n = 0, \dots, N, \quad y_0 = x_0,$$

where y_n is an approximation to $x(t_n)$, $t_n = nh$, and $h > 0$ is the time step. The integer N is chosen such that $Nh = T$.

a)

Suppose the time step h is chosen so that $1 - hK > 0$. Given the assumptions (2) and (3), prove that the explicit Euler solutions y_n always satisfy

$$0 \leq y_n \leq 1, \quad n = 0, \dots, N.$$

b)

Use the compactness method to prove that the explicit Euler method converges to the unique solution of the differential equation (1).

c)

Implement the explicit Euler method and a second order Runge-Kutta method (Heun's method). Attach the computer code to your answer. Test the methods on $x' = -x^3$ with initial value $x(0) = 1$. Use $h = 0.2$, $N = 5$, so that $T = Nh = 1$. Show the exact solution and the numerical solutions in the same plot.

d)

The numerical methods implemented in **c)** can be applied without the assumption (2). Test the methods on the differential equation $x' = x^2$ with initial value $x(0) = 1$. Use $T = 0.8$, $N = 5$, $h = 0.16$. Show the exact solution and the numerical solutions in the same plot. Compute the errors e_{euler} , e_{rk} between exact and numerical solutions at $T = 0.8$.

Explain what you observe, for example by making use of the local truncation error for the Euler method. What happens with the error if $T = 0.9$, $N = 5$, $h = 0.18$.