

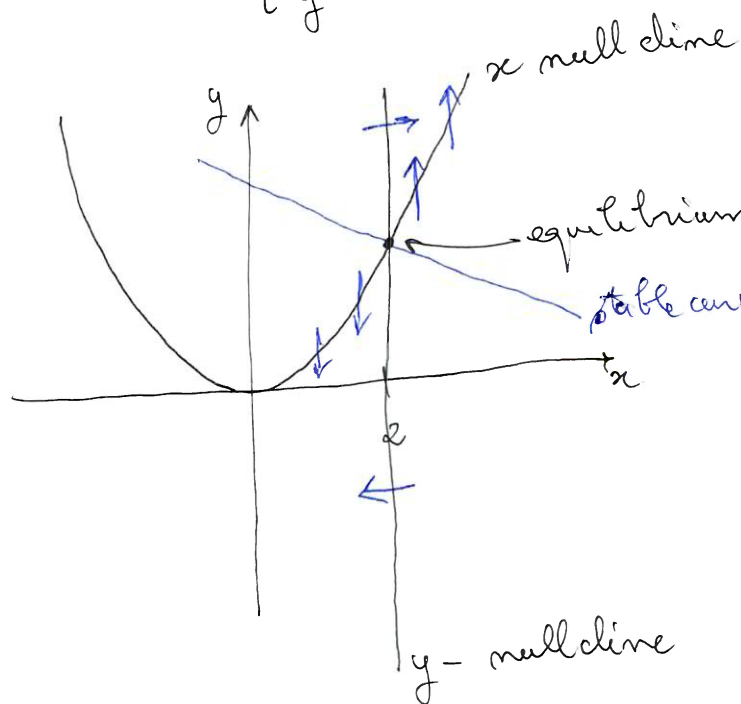
§ 9.1. Nullclines.

Chapter 9

$$\begin{cases} x_1' = f_1(x_1, \dots, x_m) \\ \vdots \\ x_m' = f_m(x_1, \dots, x_m) \end{cases}$$

x_j null-cline : points where $f_j(x_1, \dots, x_m) = 0$
 hypersurface (dimension $(m-1)$)
 on this hypersurface X' is in plane $\perp e_j$

Example
$$\begin{cases} x' = y - x^2 \\ y' = x - 2 \end{cases}$$



$$DF(x,y) = \begin{bmatrix} -2x & 1 \\ 1 & 0 \end{bmatrix} \quad \dots \quad x=2$$

$$\text{eig} : -2 \pm \sqrt{5}$$

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Example: heteroclinic bifurcation.

$$\begin{cases} x' = x^2 - 1 \\ y' = -xy + a(x^2 - 1) \end{cases}$$

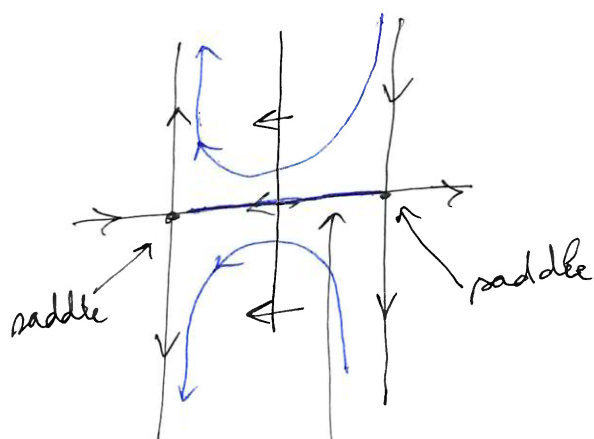
x nullclines: $x = \pm 1$

$$\begin{aligned} y' &= -y & x &= 1 \\ y' &= y & x &= -1 \end{aligned}$$

equilibrium: $(x, y) = (\pm 1, 0)$

case $a = 0$ $\begin{cases} x' = x^2 - 1 \\ y' = -xy \end{cases}$

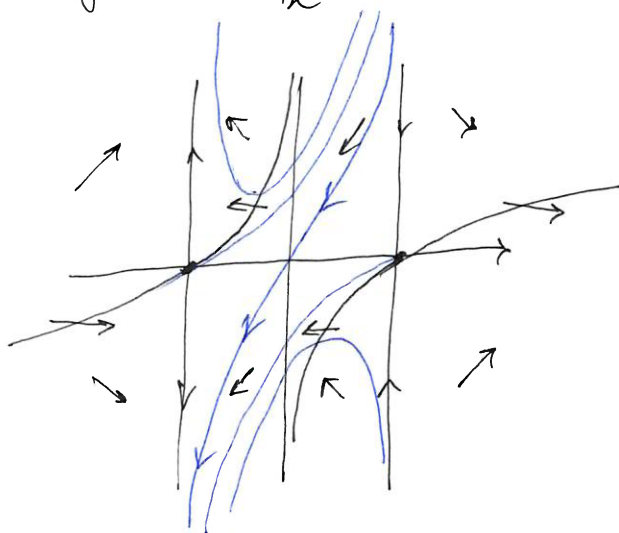
$\hookrightarrow$ nullcline: both axes



heteroclinic solution
or saddle connection

case $a \neq 0$ x -nullcline same.

y -nullcline: $y = a \frac{(x^2 - 1)}{x}$



$a > 0$

F a vector field $\leadsto X' = F(X)$

Liapunov

$L: \mathcal{O} \rightarrow \mathbb{R}$ diff. on open set

def: $\dot{L}(X) = DL(X)[F(X)]$ defined on $\mathbb{R}^n$.

remark: ϕ_t flow. Then $\frac{d}{dt}(L(\phi_t(x))) = \dot{L}(\phi_t(x))$

Theorem (and def.) X^* equilibrium. If

- a) $L(X^*) = 0$ $L(X) > 0$ for $X \neq X^*$ (strict minimum)
b) $\dot{L}(X) \leq 0$ $X \in \mathcal{O}$

then X^* stable.

c) if also $\dot{L}(X) < 0$ for $X \in \mathcal{O} \setminus \{X^*\}$
then asymptotically stable.

Proof: 1) $\delta > 0$ $\overline{B_\delta(X^*)} \subseteq \mathcal{O}$ (closed ball)
 α min of L on $\partial B_\delta(X^*)$. Then $\alpha > 0$. (compactness)

$$\mathcal{U} = \{X \in B_\delta(X^*) : L(X) < \alpha\}$$

is an open nbh of X^* .

if $X(0) \in \mathcal{U}$ then $L(X(t)) \leq L(X(0)) < \alpha$

so $X(t)$ cannot cross $\partial B_\delta(X^*)$, ie $X(t)$ stays in

$B_\delta(X^*)$, so also in $\mathcal{U}$.

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2) Suppose $X(t) = \varphi_t(x)$ a solution and that it does not converge to X^* . t_m s.t. $X(t_m) \rightarrow Z \in B_\delta(X^*)$ $Z \neq X^*$
 Then $L(X(t_m))$ decreases strictly to $L(Z)$. $\otimes$

$$L(\varphi_s(Z)) < L(Z) \text{ for any } s > 0$$

$$L(\varphi_{t_m+s}(x)) < L(Z) \text{ for } m \text{ large enough, } \rho \text{ fixed } > 0$$

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$$L(\varphi_s(\varphi_{t_m}(x)))$$

choose t_m so that $t_m > t_m + \rho$ some given ρ

$$L(\varphi_{t_m}(x)) < L(\varphi_{t_m+\rho}(x)) < L(Z)$$

But $\otimes$

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Lyapunov's invariance principle:

$X' = F(X)$ equilibrium X^* . $L: \mathcal{U} \rightarrow \mathbb{R}$ Lyapunov (not nec. strict)

$\mathcal{S} \subseteq \mathcal{U}$ closed nbhd of X^* , positively invariant.

We suppose there is no entire solution of $X' = F(X)$ (on $] -\infty, +\infty[$) inside $\mathcal{S}$ on which L is constant.

Then for $X(0) \in \mathcal{S}$ $X(t) \rightarrow X^*$: asymptotically stable.

Proof: If not: $t_n \rightarrow +\infty$ s.t. $X(t_n) \rightarrow Z \neq X^*$.

$\phi_t(Z)$ defined for all $t \in \mathbb{R}$: $\forall t \geq 0$ ok.

2) $\phi_t(X(t_n))$ defined on $[-t_n, 0]$ so also $\phi_t(X(t_{n+k}))$.

let $k \rightarrow +\infty$: $\phi_t(Z)$ defined on $[-t_n, 0]$ any $n \in \mathbb{N}$.

so $\phi_t(Z)$ defined on $] -\infty, 0]$.

remark: $L(X(t_n))$ decreases to $L(Z)$

for any $s_n \rightarrow +\infty$ $L(X(s_n))$ tends to $L(Z)$ (monotonicity)

$L(\phi_t(Z)) = L(\phi_t(X(t_n))) \rightarrow L(Z)$ so $L(\phi_t(Z)) = L(Z)$

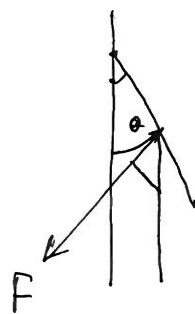
ie L is constant on the entire solution $\phi_t(Z)$ $t \in \mathbb{R}$

Non-linear pendulum

$$ml\theta'' + b\ell\theta' + mg\sin\theta = 0$$

$$\theta'' + b\theta' + \sin\theta = 0$$

$$\begin{aligned} m &= 1 \\ \ell &= 1 \\ g &= 1 \end{aligned}$$



$$v = \theta' \text{ gives: } \begin{cases} v' = -bv - \sin\theta \\ \theta' = v \end{cases}$$

$$H(v, \theta) = \frac{1}{2}v^2 - \cos\theta + 1$$

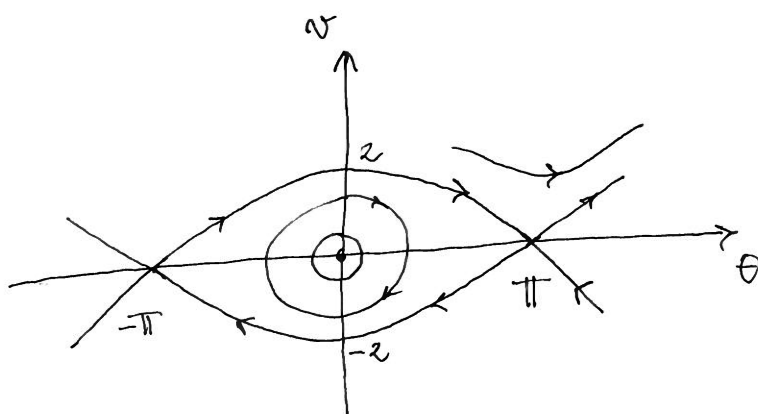
minimum 0 at $v=0, \theta = 2\pi k, k \in \mathbb{Z}$.

$$\begin{cases} v' = -\frac{\partial H}{\partial \theta} - bv \\ \theta' = \frac{\partial H}{\partial v} \end{cases}$$

linearization at $(0,0)$: $\begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$
not hyperbolic

$$\dot{H}(\theta, v) = -bv^2 \leq 0.$$

Lieapunov but not strict since $\dot{H}(0, \theta) = 0$ any θ
(n.b. choice of axis!)



$$\text{For } c < 2 \quad \mathcal{B}_c = \{(\theta, v) : |\theta| < \pi \text{ and } H(\theta, v) \leq c\}$$

• is positively invariant ($|\theta| < \pi$) if $\theta(t) \geq \pi$ some $\theta(t_0) = \pi$
the $H(v(t_0), \theta(t_0)) \geq 2$ (case $b > 0$)
• consider an entire trajectory on which H is constant

then $v=0$ and $\sin\theta=0$ so $\theta=0$

no $\mathcal{B}_c$ is basin of attraction for $(0,0)$

no $\bigcup_{c < 2} \mathcal{B}_c$ is a basin of attraction

study: case, $c = 2$

$c > 2$ unbounded solutions

linearization at $(0, \pi)$ is a saddle

ω -limit point:

$X(t)$ defined for $t \geq 0$

set of points that are limits of $X(t_n)$ $t_n \rightarrow +\infty$
constitute a closed set invariant under the flow

compare: $\bigcap_{t \geq 0} \overline{X([t, +\infty[)}$

ingibj@math.uio.no

Gradient system:

$$\langle \underset{\text{column vector}}{\text{grad } V(X)}, \underset{\text{line vector}}{\xi} \rangle = DV(X) \begin{bmatrix} \xi \\ \xi \end{bmatrix}$$

$(\mathbb{R}^n)^*$

$$X' = -\text{grad } V(X)$$

$$\dot{V}(X) = DV(X)[F(X)] = -|\text{grad } V(X)|^2$$

$$\frac{d}{dt} V(\phi_t(X)) = \dot{V}(\phi_t(X))$$

Prop: V strictly decreasing along any non constant solution.

- $\dot{V}(X) = 0 \Leftrightarrow X$ eq. point
- X^* minimum and isolated critical point $\Rightarrow$ asympt. stable.

Remark: Level set $V^{-1}(\{c\})$ $c \in \mathbb{R}$:

$X \in V^{-1}(\{c\})$ regular means $\text{grad } V(X) \neq 0$

At regular points: hypersurface and $\text{grad } V(X)$ orthogonal (usual scalar) $\uparrow$ impl. func. th.

Prop: If X^* is an ω -limit point then X^* is an equilibrium.

Proof: see LaSalle.

Remark: linearization at X^* : $D^2V(X^*)$ is symmetric hence all eigenvalues are real.

Hamiltonian systems

$$\begin{cases} x' = \frac{\partial H}{\partial y}(x, y) \\ y' = -\frac{\partial H}{\partial x}(x, y) \end{cases}$$

$$X' = J DH(X)$$

linearization at (x^*, y^*) :

$$\begin{bmatrix} \frac{\partial^2 H}{\partial x \partial y} & \frac{\partial^2 H}{\partial y \partial y} \\ -\frac{\partial^2 H}{\partial x \partial x} & -\frac{\partial^2 H}{\partial x \partial y} \end{bmatrix}$$

trace 0!

characteristic equation:

$$(\lambda - a)(\lambda + a) + bc = \lambda^2 - a^2 + bc$$

$$\text{roots } \pm \sqrt{bc - a^2} \quad \text{or} \quad \pm i \sqrt{bc - a^2}$$

depending on sign

real: saddle

circles.

at a minimum of H the

matrix $\begin{bmatrix} \frac{\partial^2}{\partial x \partial x} & \frac{\partial^2}{\partial x \partial y} \\ \frac{\partial^2}{\partial x \partial y} & \frac{\partial^2}{\partial y \partial y} \end{bmatrix}$ is $\gg 0$

so det is $\gg 0$, $bc - a^2 \gg 0$,

so roots are imaginary.

