

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT3440 — Dynamical systems

Day of examination: Friday 12, June 2026

Examination hours: 09:00–13:00

This problem set consists of 3 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Please make sure to justify all answers.

Problem 1

We consider the family of planar systems

$$X'(t) = \begin{bmatrix} \alpha & 1 \\ -1 & \alpha \end{bmatrix} X(t) \quad t \in \mathbb{R},$$

parameterized over $\alpha \in \mathbb{R}$.

1a (weight 10%)

Classify the equilibrium point $X^* = (0, 0)$ for all values of α , and **describe** the bifurcation point.

1b (weight 10%)

Sketch phase portraits in a region near $(0, 0)$ for the three systems $\alpha = -1$, $\alpha = 0$, and $\alpha = 1$.

Problem 2

We consider the initial value problem:

$$\begin{aligned} x'(t) &= e^{-t} x^2 \\ x(0) &= x_0 \in \mathbb{R}. \end{aligned} \tag{1}$$

2a (weight 10%)

Explain why there exists a locally unique C^1 -solution to (1) for all times near $t = 0$ and **compute the solution of this problem**.

(Continued on page 2.)

2b (weight 15%)

Let $I_{x_0} \subset \mathbb{R}$ denote maximal open interval on which the unique C^1 -solution to (1) is defined for a given $x_0 \in \mathbb{R}$.

- (i) **Show that** if $x_0 \in [0, 1]$, then $I_{x_0} = \mathbb{R}$.
- (ii) **Compute** $I_{x_0} = (\alpha, \infty)$ when $x_0 < 0$ **and** $I_{x_0} = (-\infty, \beta)$ when $x_0 > 1$.

Problem 3

We consider the autonomous nonlinear ODE

$$\begin{aligned} x' &= x(1 - x^2 - y^2) - y \\ y' &= y(1 - x^2 - y^2) + x \end{aligned} \tag{2}$$

For points in $\mathbb{R}^2$, we also employ the notation $X := (x, y)$ and $|X| := \sqrt{x^2 + y^2}$.

3a (weight 10%)

Show that $(0, 0)$ is the only equilibrium point of (2) **and classify** this equilibrium point.

3b (weight 10%)

Show that for any $a \in (0, 1)$ and $b > 1$, the annulus $D_{a,b} = \{X \in \mathbb{R}^2 \mid a \leq |X| \leq b\}$ is positively invariant.

3c (weight 10%)

Explain why (2) has at least one periodic solution **and compute** the $\omega(X)$ -limit set for any $X \in \mathbb{R}^2$.

Problem 4

In this problem we will solve the initial value problem

$$\begin{aligned} X' &= \underbrace{\begin{bmatrix} -4 & 2 \\ 2 & -4 \end{bmatrix}}_{=A} X \\ X(0) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned} \tag{3}$$

numerically with the Crank–Nicolson method

$$X_{n+1} = X_n + \frac{h}{2}(F(t_n, X_n) + F(t_{n+1}, X_{n+1})).$$

4a (weight 10%)

Using the time step $h = 1$, **compute** the numerical solution X_1 .

(Continued on page 3.)

4b (weight 5%)

Show that the stability function of the Crank–Nicolson method is given by

$$R(z) = \frac{1 + z/2}{1 - z/2}$$

(this relates to solving Dahlquist’s test problem $x' = \lambda x$ for $\lambda \in \mathbb{C}$).

4c (weight 10%)

Find the maximal value $h_{\max} \in (0, \infty]$ such that for any uniform time step $h < h_{\max}$, the numerical solution of the initial value problem (3) satisfies

$$|X_n| \leq C \quad \forall n \geq 0,$$

for some constant $C > 0$ that you do not need to determine.

Hint: This relates to the eigenvalues of A .

THE END