

# UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT3440 — Dynamical systems

Day of examination: Thursday 13, August 2026

Examination hours: 09:00–13:00

This problem set consists of 6 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

**Please make sure to justify all answers.**

## Problem 1

We consider the linear ODE

$$X'(t) = \begin{bmatrix} 2 & -1 \\ 1 & 3 \end{bmatrix} X \quad (1)$$

**1a** (weight 5%)

Find a similarity transformation  $T \in \mathbb{R}^{2 \times 2}$  such that

$$T^{-1}AT = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

for some  $\alpha, \beta \in \mathbb{R}$ .

**1b** (weight 10%)

Find the general solution of the linear ODE.

**1c** (weight 5%)

Classify the equilibrium point  $(0, 0)$ .

### **Solution suggestion: 1a) Similarity transformation**

The characteristic equation is  $\det(A - \lambda I) = \lambda^2 - 5\lambda + 7$ , with solutions  $\lambda_{1,2} = \frac{5 \pm i\sqrt{3}}{2} =: \alpha \pm i\beta$ , with  $\alpha = \frac{5}{2}$  and  $\beta = \frac{\sqrt{3}}{2}$ . An eigenvector for  $\lambda_1$  solves

$$(A - \lambda I)V = \begin{bmatrix} -1/2 - i\beta & -1 \\ 1 & 1/2 - i\beta \end{bmatrix} v = 0, \quad \text{with solution} \quad V = \begin{bmatrix} 1 \\ -1/2 \end{bmatrix} + i \begin{bmatrix} 0 \\ -\beta \end{bmatrix} =: V_1 + iV_2.$$

(Continued on page 2.)

We see that  $V_1$  and  $V_2$  are linearly independent and setting  $T := [V_1 \ V_2]$  yields

$$AT = T \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}.$$

This shows that  $T$  is a suitable transformation matrix.

**1b)** If  $Y$  solves

$$Y' = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} Y,$$

then  $X := TY$  solves  $X' = AX$ .

The general solution to the ODE for  $Y$  with a Jordan block matrix is, using the formula treated in the course,

$$Y(t; c) = c_1 e^{\alpha t} \begin{bmatrix} \cos(\beta t) \\ \sin(-\beta t) \end{bmatrix} + c_2 e^{\alpha t} \begin{bmatrix} \sin(\beta t) \\ \cos(\beta t) \end{bmatrix} \quad c_1, c_2 \in \mathbb{R}.$$

And the related general solution to the ODE (1) becomes

$$\begin{aligned} X(t) &= TY(t; c) = y_1(t; c)V_1 + y_2(t; c)V_2 \\ &= e^{\alpha t}(c_1 \cos \beta t + c_2 \sin \beta t) \begin{bmatrix} 1 \\ -1/2 \end{bmatrix} + e^{\alpha t}(c_1 \sin(-\beta t) + c_2 \cos \beta t) \begin{bmatrix} 0 \\ -\beta \end{bmatrix} \end{aligned}$$

**1c)**

The complex-valued eigenvalues for  $A$  have a strictly positive real part  $\alpha > 0$ . The equilibrium point is therefore unstable (and it is a spiral source since  $\beta \neq 0$ ).

## Problem 2 (weight 20%)

We consider the following family of autonomous ODEs

$$x' = -x(x^2 - \alpha)(x^2 + 4\alpha)$$

parametrized by  $\alpha \in \mathbb{R}$ . Determine and classify the equilibrium points of the ODE for every value of  $\alpha$  and draw a bifurcation diagram.

**Solution suggestion:** The equilibria are the subset of the zeros of  $f(x) := -x(x^2 - \alpha)(x^2 + 4\alpha)$ :  $x = 0$ ,  $x^2 = \alpha$ ,  $x^2 = -4\alpha$  that are real-valued.

The number and location of EP (real-valued zeros of  $f$ ) depend on the sign of  $\alpha$ :

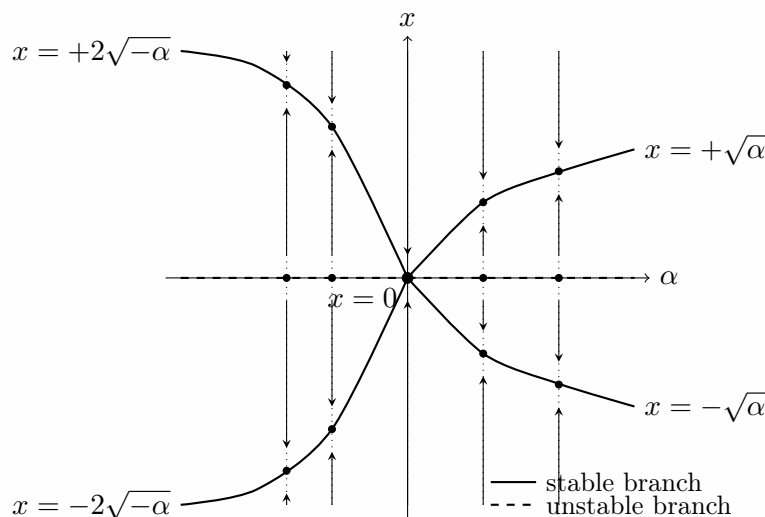
- $\alpha > 0$ :  $x = 0$ ,  $x = \pm\sqrt{\alpha}$ .
- $\alpha = 0$ :  $x = 0$ .
- $\alpha < 0$ :  $x = 0$ ,  $x = \pm 2\sqrt{-\alpha}$ .

Stability of EP:

- $\alpha > 0$ :  $f'(0) = 4\alpha^2 > 0 \implies x = 0$  is unstable and  $f'(\pm\sqrt{\alpha}) = -10\alpha^2 < 0 \implies \pm\sqrt{\alpha}$  are asymptotically stable.

(Continued on page 3.)

- $\alpha = 0$ :  $f(x) = -x^5$ . So  $f > 0$  for  $x < 0$  and  $f < 0$  for  $x > 0$ , which implies that 0 is asymptotically stable (solve e.g. by separation of variables or draw phase line to show this property).
- $\alpha < 0$ :  $f'(0) = 4\alpha^2 > 0 \implies x = 0$  is unstable and  $f'(\pm 2\sqrt{-\alpha}) = -40\alpha^2 < 0 \implies \pm 2\sqrt{-\alpha}$  are asymptotically stable.



### Problem 3

For the potential function  $V: \mathbb{R}^2 \rightarrow \mathbb{R}$  defined by

$$V(x, y) := \frac{1}{2}(y - x^2)^2 + \frac{1}{6}x^4(1 - x^2) + \frac{1}{12}x^4,$$

we consider the gradient flow

$$\begin{aligned} x' &= -V_x(x, y) \\ y' &= -V_y(x, y). \end{aligned} \tag{2}$$

#### 3a (weight 15%)

Find all equilibrium points of the dynamical system. Determine which of the equilibrium points can be classified by the linearization method, and classify those equilibrium points.

#### 3b (weight 10%)

Use Lyapunov theory to show that  $(0, 0)$  is an asymptotically stable equilibrium point.

**Solution suggestion: 3a:** The dynamical system can be written

$$\begin{aligned} x' &= 2x(y - x^2) - x^3 + x^5 \\ y' &= -(y - x^2) \end{aligned}$$

(Continued on page 4.)

The second equation of  $-\nabla V = 0$  tells us that  $y = x^2$  at an equilibrium point. Substituting this into the first equation yields  $x^3(x^2 - 1) = 0$ . Hence we have three equilibrium points

$$X_0^* = (0, 0), \quad X_1^* = (-1, 1), \quad X_2^* = (1, 1)$$

Let  $F(X) := -\nabla V(X)$  and note that

$$DF(X) = \begin{bmatrix} 2y - 9x^2 + 5x^4 & 2x \\ 2x & -1 \end{bmatrix}.$$

Since

$$DF(X_0^*) = \begin{bmatrix} 0 & 0 \\ 0 & -1 \end{bmatrix},$$

with eigenvalues  $\lambda_1 = 0$  and  $\lambda_2 = -1$ ,  $DF(X_0^*)$  is not a hyperbolic matrix, so we cannot apply the linearization theorem to classify  $X_0^*$ .

Next,

$$DF(X_1^*) = \begin{bmatrix} -2 & -2 \\ -2 & -1 \end{bmatrix},$$

with eigenvalues  $\lambda_{\pm} = \frac{-3 \pm \sqrt{17}}{2}$ , so  $DF(X_1^*)$  is a hyperbolic matrix, the linearization theorem applies and  $X_1^*$  is a saddle (unstable).

Last,

$$DF(X_2^*) = \begin{bmatrix} -2 & 2 \\ 2 & -1 \end{bmatrix},$$

with the same eigenvalues  $\lambda_{\pm} = \frac{-3 \pm \sqrt{17}}{2}$ , so  $X_2^*$  is also a saddle (unstable).

**3b:** We will verify that  $V$  is a strict Lyapunov function for 0. Lyapunov theory then tells us that 0 is an asymptotically stable equilibrium point.

Observe that  $V(0) = 0$  and for a point  $(x, y) \in D := B_1(0) \setminus \{0\}$  we have two possible cases: Either  $x \neq 0$  and then  $V(x, y) \geq \frac{1}{12}x^4 > 0$ , or  $x = 0$  and  $y \in (-1, 1) \setminus \{0\}$ , and then  $V(x, y) = y^2/2 > 0$ . So  $V(x, y) > 0$  for any point  $(x, y) \in D \setminus \{0\}$ . Furthermore,

$$\dot{V}(x, y) = -|\nabla V(x, y)|^2 \leq 0$$

From 3a), we know that  $(0, 0)$  is the only zero of  $\nabla V(x, y)$  in  $B_1(0)$ , so

$$\dot{V}(x, y) = -|\nabla V(x, y)|^2 < 0 \quad \forall (x, y) \in B_1(0) \setminus \{0\}.$$

## Problem 4

We consider the nonlinear ODE

$$\begin{aligned} x_1' &= -x_1 x_2^2, \\ x_2' &= -2x_2. \end{aligned} \tag{3}$$

(Continued on page 5.)

**4a** (weight 10%)

Find the exact solution to the ODE and show that for any initial condition  $X(0) = X_0 \in \mathbb{R}^2$ , it holds that

$$|X(t)| \leq |X_0| \quad t \geq 0.$$

**4b** (weight 10%)

Recall that the forward Euler method is given by

$$X_{n+1} = X_n + hF(t_n, X_n)$$

Use the forward Euler method with time step  $h = 1$  to iteratively solve the initial value problem (3) with  $X_0 = (1, 2)$ . Determine  $X_n$  for  $n \geq 0$  and show that

$$\lim_{n \rightarrow \infty} |X_n| = \infty.$$

**4c** (weight 15%)

Propose another Runge–Kutta method that satisfies

$$|X_n| \leq |X_0| \quad n \geq 0$$

for any  $X_0 \in \mathbb{R}^2$  and any time step  $h > 0$ . Show that the method you propose has this property.

**Solution suggestion: 4a)** First solve the scalar linear ODE for  $x_2$ :

$$x_2' = -2x_2 \implies x_2(t) = x_{2,0}e^{-2t}.$$

Insert this into the equation for  $x_1$ :

$$x_1' = -x_1(t)(x_2(t))^2 = -x_1(t)x_{2,0}^2e^{-4t}.$$

Separation of variables yields

$$x_1(t) = x_{1,0} \exp\left(-\frac{x_{2,0}^2}{4}(1 - e^{-4t})\right)$$

The exact solution is

$$X(t) = (x_1(t), x_2(t)) = \left(x_{1,0} e^{-\frac{x_{2,0}^2}{4}(1 - e^{-4t})}, x_{2,0} e^{-2t}\right)$$

and for all  $t \geq 0$ ,

$$|X(t)|^2 = x_{1,0}^2 e^{-\frac{x_{2,0}^2}{2}(1 - e^{-4t})} + x_{2,0}^2 e^{-4t} \leq |X_0|^2.$$

**4b)** Forward Euler:

$$X_{n+1} = X_n + hF(X_n) = \begin{bmatrix} x_{1,n} - hx_{1,n}x_{2,n}^2 \\ x_{2,n} - 2hx_{2,n} \end{bmatrix}$$

(Continued on page 6.)

For  $h = 1$  we get

$$x_{1,n+1} = x_{1,n} - x_{1,n}x_{2,n}^2 = x_{1,n}(1 - x_{2,n}^2),$$

$$x_{2,n+1} = x_{2,n} - 2x_{2,n} = -x_{2,n}$$

With  $X_0 = (1, 2)$ , we get  $x_{2,n} = (-1)^n 2$  and

$$x_{1,n+1} = x_{1,n}(1 - x_{2,n}^2) = -3x_{1,n} \implies x_{1,n} = (-3)^n x_{1,0} = (-3)^n$$

Hence,

$$|X_n|^2 = x_{1,n}^2 + x_{2,n}^2 = 9^n + 4 \rightarrow \infty \quad \text{as } n \rightarrow \infty.$$

**4c)**

Consider the implicit Euler method

$$X_{n+1} = X_n + hF(X_{n+1}), \quad h > 0.$$

Applied to (3), this gives the nonlinear system

$$x_{1,n+1} = x_{1,n} - hx_{1,n+1}x_{2,n+1}^2,$$

$$x_{2,n+1} = x_{2,n} - 2hx_{2,n+1}.$$

This can be solved iteratively by the steps

$$x_{2,n+1} = \frac{x_{2,n}}{1 + 2h}$$

and

$$x_{1,n+1} = \frac{x_{1,n}}{1 + hx_{2,n+1}^2}.$$

We conclude that  $|x_{1,n+1}| \leq |x_{1,n}|$  and  $|x_{2,n+1}| \leq |x_{2,n}|$ , so that

$$|X_{n+1}|^2 = x_{1,n+1}^2 + x_{2,n+1}^2 \leq |X_n|^2 \implies |X_n|^2 \leq |X_0|^2 \quad \forall n \geq 0.$$

THE END