

UNIVERSITY OF OSLO

Faculty of mathematics and natural sciences

Exam in: MAT3440 — Dynamical systems

Day of examination: Friday 12, June 2026

Examination hours: 09:00–13:00

This problem set consists of 6 pages.

Appendices: None

Permitted aids: None

Please make sure that your copy of the problem set is complete before you attempt to answer anything.

Please make sure to justify all answers.

Problem 1

We consider the family of planar systems

$$X'(t) = \begin{bmatrix} \alpha & 1 \\ -1 & \alpha \end{bmatrix} X(t) \quad t \in \mathbb{R},$$

parameterized over $\alpha \in \mathbb{R}$.

1a (weight 10%)

Classify the equilibrium point $X^* = (0, 0)$ for all values of α , and **describe** the bifurcation point.

1b (weight 10%)

Sketch phase portraits in a region near $(0, 0)$ for the three systems $\alpha = -1$, $\alpha = 0$, and $\alpha = 1$.

Solution suggestion: 1a: The characteristic equation for the matrix $\begin{bmatrix} \alpha & 1 \\ -1 & \alpha \end{bmatrix}$ is $\lambda^2 - 2\alpha\lambda + (1 + \alpha^2) = 0$. This yields the complex-valued eigenvalues $\lambda_{\pm} = \alpha \pm i$. For $\alpha < 0$, $X^* = 0$ is a spiral sink (asymptotically stable), for $\alpha = 0$, $X^* = 0$ is a center (stable) and for $\alpha > 0$, $X^* = 0$ is a spiral source (unstable).

We see that $\alpha = 0$ is a bifurcation point where the equilibrium $X^*(\alpha)$ changes from a spiral sink to a spiral source via a center.

1b:

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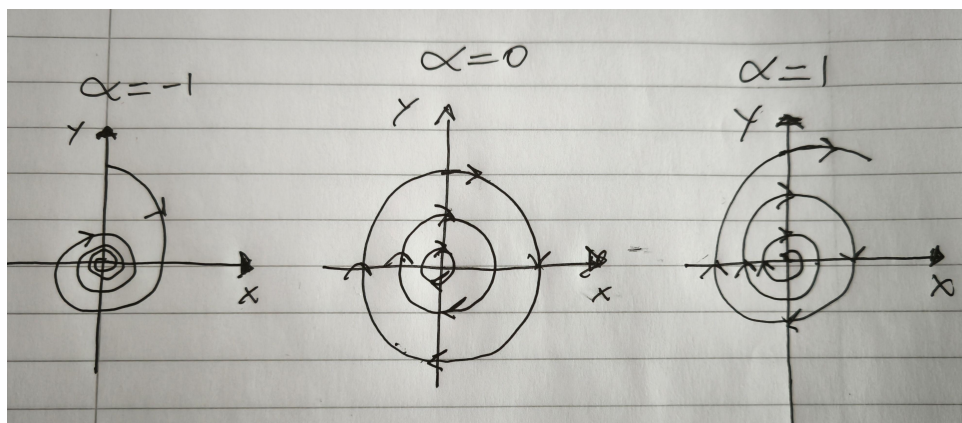


Figure 1: Phase portraits for Problem 1b.

Problem 2

We consider the initial value problem:

$$\begin{aligned} x'(t) &= e^{-t}x^2 \\ x(0) &= x_0 \in \mathbb{R}. \end{aligned} \tag{1}$$

2a (weight 10%)

Explain why there exists a locally unique C^1 -solution to (1) for all times near $t = 0$ and **compute the solution of this problem.**

2b (weight 15%)

Let $I_{x_0} \subset \mathbb{R}$ denote maximal open interval on which the unique C^1 -solution to (1) is defined for a given $x_0 \in \mathbb{R}$.

- (i) **Show that** if $x_0 \in [0, 1]$, then $I_{x_0} = \mathbb{R}$.
- (ii) **Compute** $I_{x_0} = (\alpha, \infty)$ when $x_0 < 0$ and $I_{x_0} = (-\infty, \beta)$ when $x_0 > 1$.

Solution suggestion: 2a: Since the function $f(x, t) := e^{-t}x^2$ is a smooth function: $f \in C^\infty(\mathbb{R} \times \mathbb{R}, \mathbb{R})$, we have for any given $x_0 \in \mathbb{R}$ that the Picard-Lindelöf theorem ensures local existence and uniqueness of a C^1 -solution to (1) in an open interval containing $t = 0$.

If $x_0 = 0$, we see directly that $x(t) = 0$ for all $t \in \mathbb{R}$ is a solution. By maximal extension of the locally unique solution, this is the globally unique solution for $x_0 = 0$.

When $x_0 \neq 0$, separation of variables yields the local solution

$$\int x^{-2} dx = \int e^{-t} dt \implies x(t) = \frac{1}{e^{-t} - 1 + x_0^{-1}}.$$

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2b: From the theory of maximal solutions we know since f is smooth that the unique solution can be extended as long as $x(t)$ remains finite. In this case the domain of $f(t, x)$ is all of $\mathbb{R}^2$, so the only obstruction to further extension is that $|x(t)| \rightarrow \infty$ at some finite time.

Writing the solution as follows

$$x(t) = \frac{x_0}{(e^{-t} - 1)x_0 + 1},$$

we see that if $x_0 \in (0, 1]$, then $(e^{-t} - 1)x_0 + 1 > 0$ for all $t \in \mathbb{R}$, which implies that $x(t) \in \mathbb{R}$ for all $t \in \mathbb{R}$. And if $x_0 = 0$, the unique solution $x(t) = 0$ clearly also is defined for all $t \in \mathbb{R}$.

If $x_0 < 0$, then the solution is well-defined until the first time when $(e^{-t} - 1)x_0 + 1 = 0$. The only time when this occurs is at $\alpha = \log(x_0/(x_0 - 1)) < 0$ and $\lim_{t \downarrow \alpha} x(t) = -\infty$. We conclude that $I_{x_0} = (\alpha, \infty)$.

By similar reasoning, we also have for $x_0 > 1$ that the only time when $(e^{-t} - 1)x_0 + 1 = 0$ is $\beta = \log(x_0/(x_0 - 1)) > 0$ and $\lim_{t \uparrow \beta} x(t) = \infty$. We conclude that $I_{x_0} = (-\infty, \beta)$.

Problem 3

We consider the autonomous nonlinear ODE

$$\begin{aligned} x' &= x(1 - x^2 - y^2) - y \\ y' &= y(1 - x^2 - y^2) + x \end{aligned} \tag{2}$$

For points in $\mathbb{R}^2$, we also employ the notation $X := (x, y)$ and $|X| := \sqrt{x^2 + y^2}$.

3a (weight 10%)

Show that $(0, 0)$ is the only equilibrium point of (2) **and classify** this equilibrium point.

3b (weight 10%)

Show that for any $a \in (0, 1)$ and $b > 1$, the annulus $D_{a,b} = \{X \in \mathbb{R}^2 \mid a \leq |X| \leq b\}$ is positively invariant.

3c (weight 10%)

Explain why (2) has at least one periodic solution **and compute** the $\omega(X)$ -limit set for any $X \in \mathbb{R}^2$.

Solution suggestion:

3a: If $X = (x, y)$ is an equilibrium point of (2), then the first equation of the vector field becomes $y = x(1 - x^2 - y^2)$. Substituting this into the second equation yields $x(1 + (1 - x^2 - y^2)^2) = 0$. The only solutions to the second equation are $(0, y)$ for $y \in \mathbb{R}$. Inserting these candidate solutions back into the first equation, we see that

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for $(0, y)$ that

$$y = x(1 - x^2 - y^2) \big|_{x=0} = 0,$$

so $X^* = (0, 0)$ is the only equilibrium point.

Let $F(X)$ denote the right-hand side vector field of (2). Then

$$DF(X^*) = \begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}$$

with eigenvalues $\lambda_{\pm} = 1 \pm i$. This is a hyperbolic matrix with $\text{Re}(\lambda_{\pm}) > 0$. The linearization theorem implies X^* is a spiral source (unstable).

3b: For $r^2 := x^2 + y^2$ we have that

$$\frac{d}{dt}r^2/2 = X \cdot X' = (x^2 + y^2)(1 - (x^2 + y^2)) = r^2(1 - r^2) \implies r' = r(1 - r^2).$$

Hence on the boundary of $D_{a,b}$, we have

$$r'(t) \big|_{r=a} = a(1 - a^2) > 0 \quad \text{and} \quad r'(t) \big|_{r=b} = b(1 - b^2) < 0.$$

This implies that for any $X_0 \in D_{a,b}$, it holds that $\phi_t(X_0) \in D$ for all $t \geq 0$. (Since $a \leq |\phi_t(X_0)| \leq b$ for all $t \geq 0$.)

3c: For any fixed $a \in (0, 1)$ and $b > 1$, $D_{a,b}$ is a compact, positively invariant set that contains no equilibrium points. The Poincaré-Bendixson theorem then applies, so there exists a periodic solution in $D_{a,b}$.

If $X = 0$, then $\omega(X) = \{0\}$, since this is an equilibrium point. If $|X| > 0$, then $\omega(X) = \partial B_1(0)$ by the explanation that follows.

For $|X| > 0$, the ODE $r' = r(1 - r^2)$ implies that $|X(t)| \rightarrow 1$ as $t \rightarrow \infty$ and thus that $\omega(X) \subset \partial B_1(0)$. (any point outside of the unit circle cannot be an accumulation point of $\phi_t(X)$). But for this problem, Poincaré-Bendixson also tells us that $\omega(X)$ is a closed orbit for any $|X| > 0$ (since we can choose a and b s.t. $X \in D_{a,b}$ for any $|X| > 0$), and the only closed orbit that is a subset of $\partial B_1(0)$ is the whole set itself, since $\partial B_1(0)$ itself is a simple closed curve. (Alternatively, study dynamical system in polar coordinates. Show that $\theta' = 1$ and that the dynamics for the component $r' = r(1 - r^2)$ only has one asymptotically stable EP $r = 1$. Moreover, the r -dynamics for any nonzero initial value r_0 converges to 1, meaning $\lim_{t \rightarrow \infty} r(t; r_0) = 1$ for any $r_0 = |X| > 0$.)

Problem 4

In this problem we will solve the initial value problem

$$\begin{aligned} X' &= \underbrace{\begin{bmatrix} -4 & 2 \\ 2 & -4 \end{bmatrix}}_{=A} X \\ X(0) &= \begin{bmatrix} 1 \\ 0 \end{bmatrix} \end{aligned} \tag{3}$$

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numerically with the Crank–Nicolson method

$$X_{n+1} = X_n + \frac{h}{2}(F(t_n, X_n) + F(t_{n+1}, X_{n+1})).$$

4a (weight 10%)

Using the time step $h = 1$, **compute** the numerical solution X_1 .

4b (weight 5%)

Show that the stability function of the Crank–Nicolson method is given by

$$R(z) = \frac{1 + z/2}{1 - z/2}$$

(this relates to solving Dahlquist's test problem $x' = \lambda x$ for $\lambda \in \mathbb{C}$).

4c (weight 10%)

Find the maximal value $h_{\max} \in (0, \infty]$ such that for any uniform time step $h < h_{\max}$, the numerical solution of the initial value problem (3) satisfies

$$|X_n| \leq C \quad \forall n \geq 0,$$

for some constant $C > 0$ that you do not need to determine.

Hint: This relates to the eigenvalues of A .

Solution suggestion: 4a: Let $A = \begin{bmatrix} -4 & 2 \\ 2 & -4 \end{bmatrix}$. Then

$$X_1 = X_0 + \frac{1}{2}(AX_0 + AX_1) \implies X_1 = (I - \frac{1}{2}A)^{-1}(I + \frac{1}{2}A)X_0 = \frac{1}{8} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix} \begin{bmatrix} -1 \\ 1 \end{bmatrix} = \frac{1}{4} \begin{bmatrix} -1 \\ 1 \end{bmatrix}$$

where we used that

$$(I - \frac{1}{2}A)^{-1} = \begin{bmatrix} 3 & -1 \\ -1 & 3 \end{bmatrix}^{-1} = \frac{1}{8} \begin{bmatrix} 3 & 1 \\ 1 & 3 \end{bmatrix}$$

and

$$(I + \frac{1}{2}A) = \begin{bmatrix} -1 & 1 \\ 1 & -1 \end{bmatrix}$$

4b: For $f(x) = \lambda x$, we have that

$$x_{n+1} = x_n + \frac{h}{2}(f(x_n) + f(x_{n+1})) = x_n + \lambda h(x_n/2 + x_{n+1}/2) \implies x_{n+1} = \frac{1 + \lambda h/2}{1 - \lambda h/2} x_n,$$

where the first factor with $z = \lambda h$ yields the stability function $R(z) = (1 + z/2)/(1 - z/2)$.

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4c: Since A is symmetric, we need to find the largest possible $h_{\max} > 0$ s.t.

$$|R(\lambda_i h)| \leq 1 \quad i = 1, 2$$

holds for any $h < h_{\max}$.

The characteristic equation for A is $\lambda^2 + 8\lambda + 12 = 0$ with $\lambda_1 = -6$ and $\lambda_2 = -2$. And $\lambda_i < 0$ for $i = 1, 2$ implies that

$$|R(\lambda_i h)| = \frac{|1 + h\lambda_i/2|}{|1 - \lambda_i h/2|} = \frac{|1 + h\lambda_i/2|}{1 + |\lambda_i h/2|} \leq 1 \quad \forall h > 0.$$

We conclude that $h_{\max} = \infty$.

(Not required in the answer but the reason why this timestep constraint holds with $C = |X_0| = 1$ is that $\|(I - Ah/2)^{-1}(I + Ah/2)\|_2 = \|R(Ah)\|_2 = \max_i |R(\lambda_i h)|$ by the spectral theorem for symmetric matrices, and

$$|X_{n+1}| \leq \|(I - Ah/2)^{-1}(I + Ah/2)\|_2 |X_n|.)$$

THE END