

lecture 30 Mat 3440

Exam 2023 Problems 1, 2, 3a, 3b, 4

Exam 2024 Utsatt All problems

Exam 2025 All problems

2023

1a Compute all equilibria and determine their stability of $x' = f_r(x) = rx + x^3$, $r \in \mathbb{R}$

Draw a bifurcation diagram. What is the bifurcation point?

Sol $x(r + x^2) = 0$

EP: $x_0 = 0$

if $r \geq 0$

$x_0 = 0, x_{\pm} = \pm\sqrt{-r}$

if $r < 0$

$f_r'(x) = r + 3x^2$, so by linearization,

x_0 is $\begin{cases} \text{source} & \text{if } r > 0 \\ \text{unstable} & \text{if } r = 0 \\ \text{sink} & \text{if } r < 0 \end{cases}$

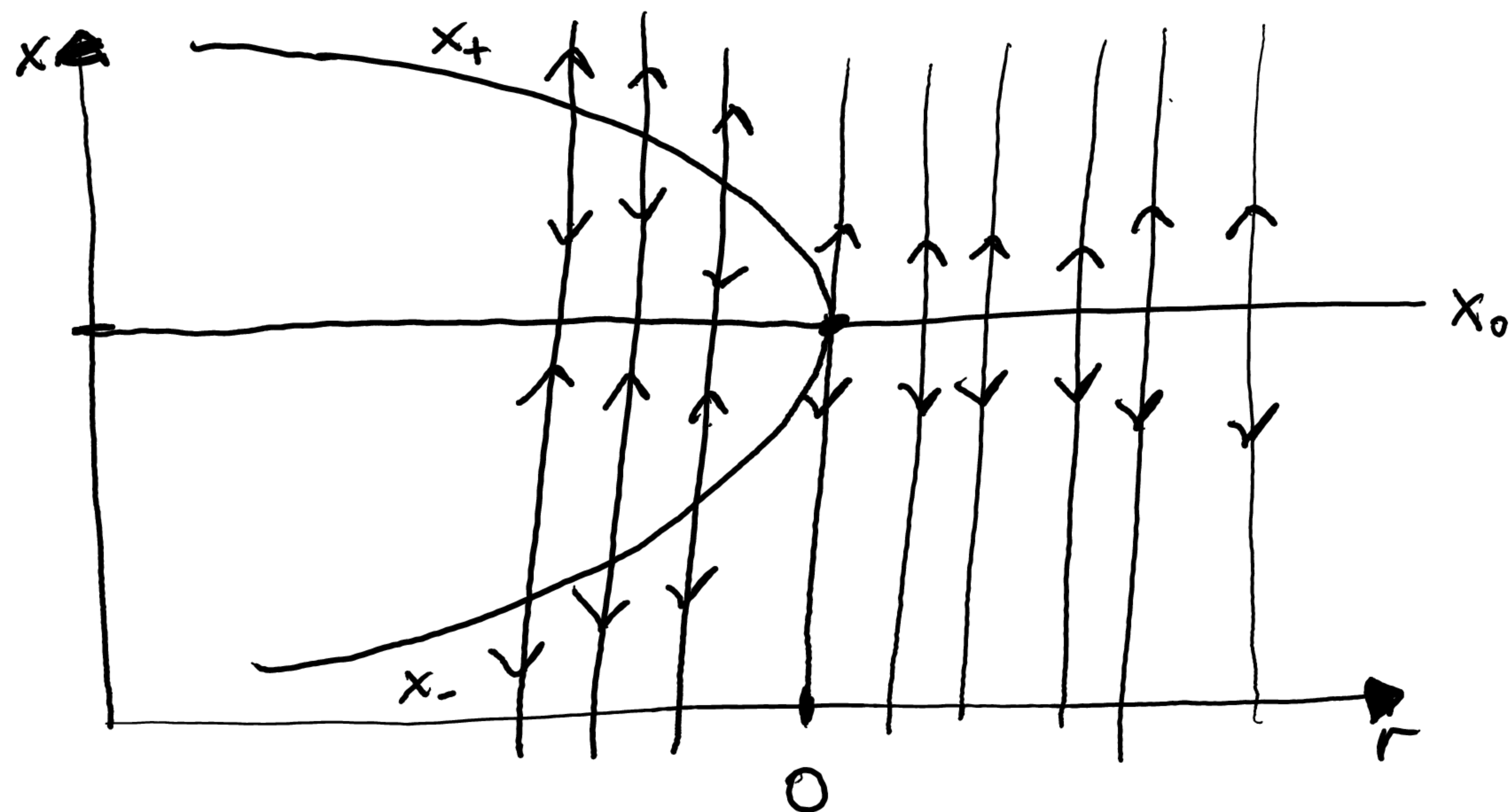
Phase line $r=0$



x_+ with $f'_r(x_+) = r - 3r = -2r > 0$

is a source

x_- with $f'_r(x_-) = -2r > 0$ is also a source.



Bifurcation point $r=0$.

(Pitchfork bifurcation)

1b Consider $x' = f(x) = \frac{1}{x}$ for $f: (0, \infty) \rightarrow \mathbb{R}$.

(i) Determine $\phi_t(x_0)$ for $x_0 > 0$

(ii) Determine the maximal solution interval $I_{x_0} = (\alpha, \infty)$ and compute $\lim_{t \rightarrow \alpha} \phi_t(x_0)$

(iii) Verify that $\Phi_{t+s}(x_0) = \Phi_t(\Phi_s(x_0))$
for $s, s+t \in I_{x_0}$

sol

(i) Separation of variables

$$\int x dx = t + C$$

$$\Rightarrow x^2 = 2t + C$$

$$x_0 > 0 \Rightarrow x = \sqrt{2t + x_0^2}$$

$$\Phi_t(x_0) = \sqrt{2t + x_0^2}$$

(ii) Solution is defined while

$x(t) \in D = (0, \infty)$. This fails when

$$t = \alpha := -\frac{x_0^2}{2}, \text{ as}$$

$$\lim_{t \downarrow \alpha} \Phi_t(x_0) = 0 \notin D.$$

$$\text{so } I_{x_0} = \left(-\frac{x_0^2}{2}, \infty\right).$$

(iii)

$$\begin{aligned}\phi_t(\phi_s(x_0)) &= \sqrt{2t + (\phi_s(x_0))^2} \\ &= \sqrt{2t + \left(\sqrt{2s + x_0^2}\right)^2} \\ &= \sqrt{2(t+s) + x_0^2} \\ &= \phi_{t+s}(x_0)\end{aligned}$$

c) construct two different solutions of the IVP $x' = x^{1/3}$, $x(0) = 0$

why does the existence and uniqueness theorem fail to apply in this case?

Sol $x(t) = 0 \forall t$ is one solution. To find another solution, let us formally solve with separation of variables:

$$\int x^{-1/3} dx = t + C$$

$$\Rightarrow x^{2/3} = \frac{2}{3}t + C \Rightarrow x = \left(\frac{2}{3}t + C\right)^{3/2}$$

Setting $C=0$ yields

$$\tilde{x}(t) = \left(\frac{2}{3}t\right)^{3/2}$$

As $\tilde{x}'(t) = \left(\frac{2}{3}t\right)^{1/2} = (\tilde{x})^{1/3}$ & $\tilde{x}(0)=0$,

this is another solution.

Picard-Lindlöf does not yield a locally unique solution near $x=0$, as $f(x)$ is not locally Lipschitz continuous there:

$$\frac{|f(h) - f(0)|}{h} = h^{-2/3} \rightarrow \infty \text{ as } h \rightarrow 0.$$

Problem 2

a. Consider

$$x' = -x + by$$

$$y' = x - y$$

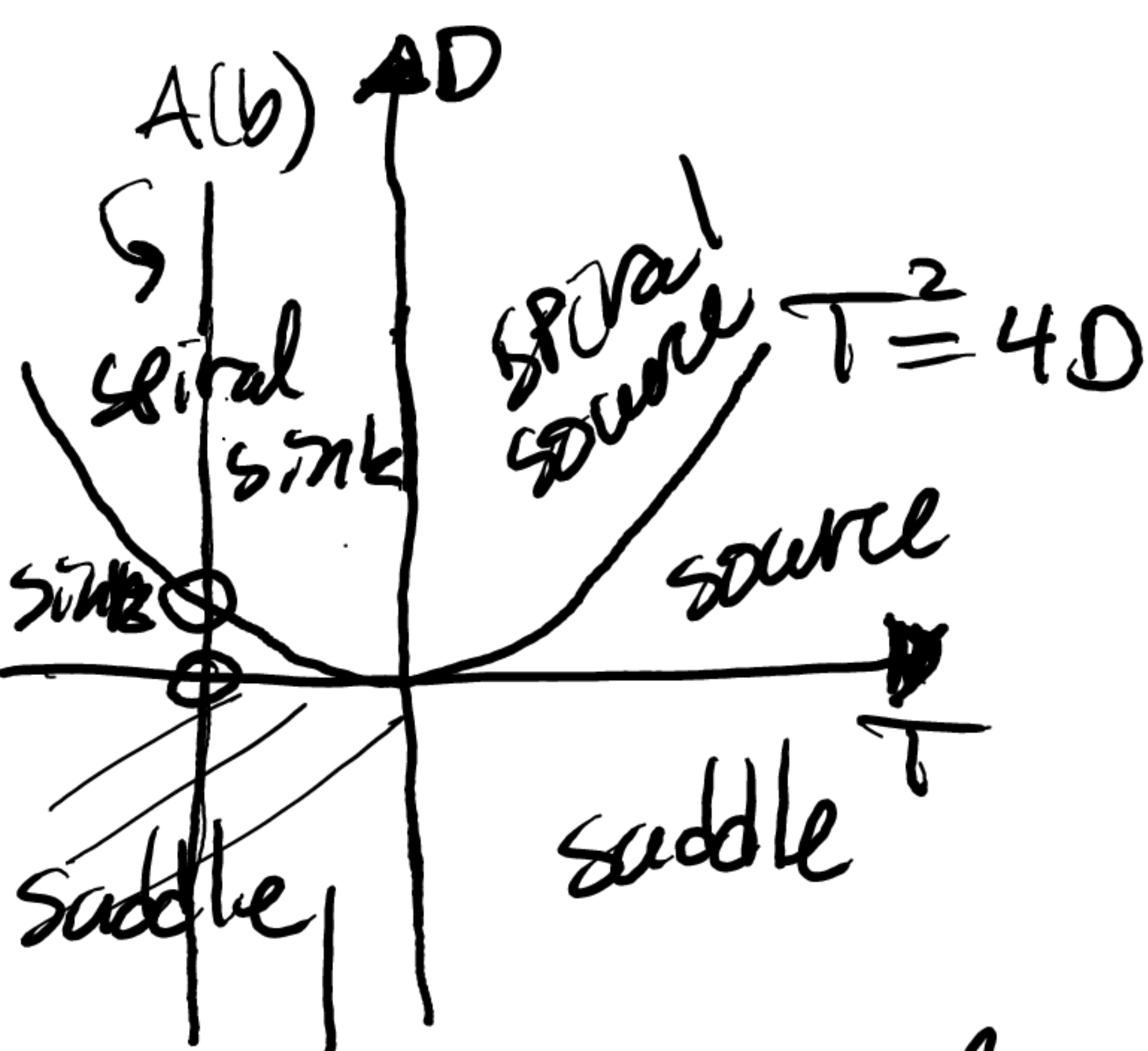
with parameter $b \in \mathbb{R}$

Use trace-determinant analysis to determine the phase portrait at 0.

sol $\bar{x}' = \begin{bmatrix} -1 & b \\ 1 & -1 \end{bmatrix} \bar{x}$

Char eq: $\lambda^2 - \text{Tr}(A)\lambda + \det(A) = 0$

$$\lambda_{\pm} = \frac{\text{Tr}(A) \pm \sqrt{\text{Tr}(A)^2 - 4\det(A)}}{2} = -1 \pm \sqrt{b}$$



$b > 0$ spiral sink

$b \in [0, 1)$ sink

$$T = -2$$

$$D = 1 - b$$

$$D(b) = \frac{T^2}{4} = 1 \text{ when } b = 0$$

$$D(b) = 0 \text{ when } b = 1$$

$b = 1$, 0 is stable
($x = x$ equilibria)

$b > 0$ 0 is saddle.

b) Consider

$$\begin{aligned}x' &= -x - y^2 \\ y' &= x - y\end{aligned} = F(\underline{x})$$

determine equilibrium points and
phase portrait near each EP.

Sol Substitute $x=y$ into first eq:

$$y(1+y) = 0$$

EP $(0,0)$, $(-1,-1)$.

$$DF = \begin{bmatrix} -1 & -2y \\ 1 & -1 \end{bmatrix}$$

$$DF(0) = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix} \quad \text{is hyperbolic}$$

Linearization then $\Rightarrow$

$$\underline{x}'(t) = \underbrace{\begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}}_A \underline{x} + O(|\underline{x}|^2) \quad \text{near } 0$$

Consider $\tilde{\underline{x}}' = A\tilde{\underline{x}}$: solution

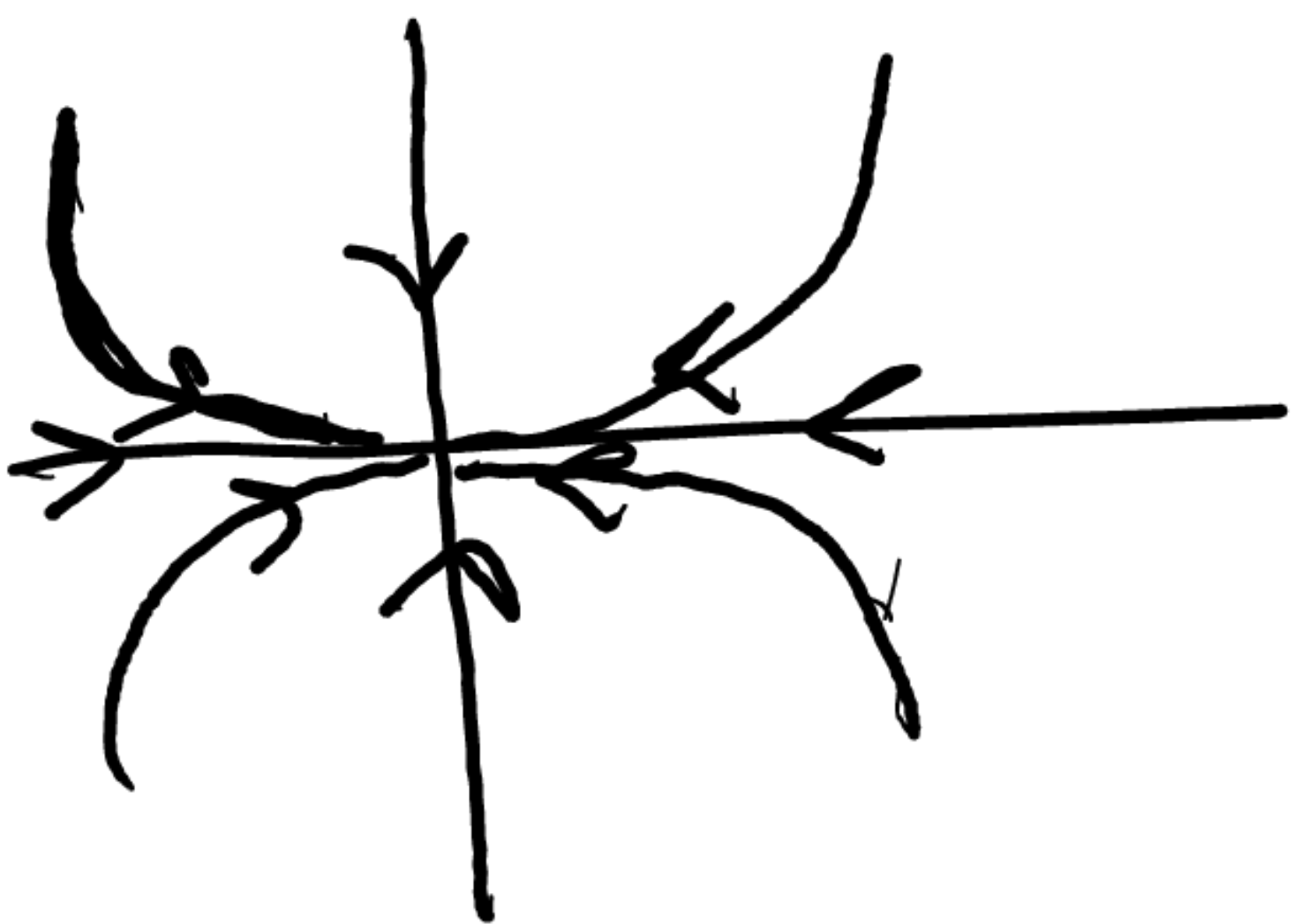
$$\tilde{x}(t) = x_0 e^{-t}$$

$$\tilde{y}'(t) = -\tilde{y} + x_0 e^{-t}$$

$$(e^t \tilde{y})' = x_0 \Rightarrow \tilde{y}(t) = e^{-t} (\gamma_0 + t x_0)$$

$$\tilde{z}(t) = \tilde{z}_0 e^{-t} + e^{-t} \begin{bmatrix} 0 \\ t x_0 \end{bmatrix} \quad (2b)$$

Phase portrait



$$EP: \tilde{z}^* = (-1, -1)$$

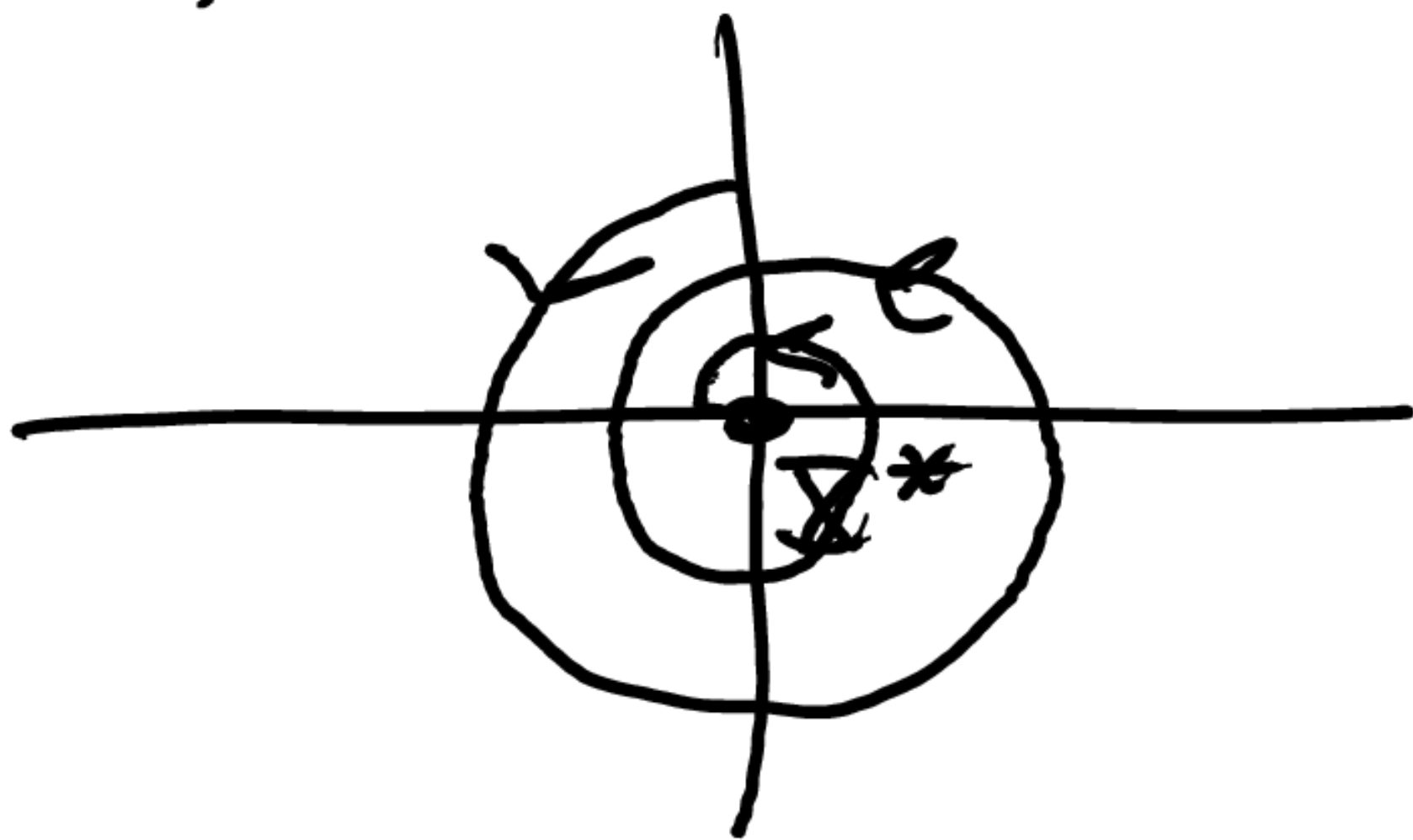
$$DF(\tilde{z}^*) = \begin{bmatrix} -1 & 2 \\ -1 & -1 \end{bmatrix}$$

$$EV: \lambda^2 + 2\lambda + 3 = 0 \quad \lambda = -1 \pm \sqrt{-8}$$

$\Rightarrow DF(\tilde{z}^*)$ is hyperbolic and

$$\tilde{z}' = DF(\tilde{z}^*)(\tilde{z} - \tilde{z}^*) + O(|\tilde{z} - \tilde{z}^*|^2)$$

is a spiral sink near $\tilde{z}^*$



c) Compute the matrix exponential
of $A = \begin{bmatrix} -1 & 0 \\ 1 & -1 \end{bmatrix}$.

Sol know that solution of
 $\dot{\tilde{x}} = A \tilde{x}$ is $\tilde{x} = e^{At} \tilde{x}_0$

$$\Rightarrow e^A = \begin{bmatrix} e^A \begin{bmatrix} 1 \\ 0 \end{bmatrix} & e^A \begin{bmatrix} 0 \\ 1 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} \tilde{x}(1; \begin{bmatrix} 1 \\ 0 \end{bmatrix}) & \tilde{x}(1; \begin{bmatrix} 0 \\ 1 \end{bmatrix}) \end{bmatrix}$$

$$\underline{\underline{(b)}} \begin{bmatrix} e^{-1} & 0 \\ e^{-1} & e^{-1} \end{bmatrix}$$

Problem 3

a)
$$\begin{aligned} x' &= y \\ y' &= -x - \varepsilon x^2 y \end{aligned} \quad \varepsilon > 0 \quad (30)$$

Explain why the Linearization Theorem cannot be used to study the EP 0:

Sol as
$$DF(0) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$$

with EV $\lambda_{\pm} = \pm i$ is not hyperbolic, the theorem does not apply.

b) (i) Formulate the Lyapunov theorem for a general $\dot{z} = F(z)$

(ii) Apply the Lyapunov theorem to determine the stability of the origin for system (3a)

so (i) Let $\bar{x}^*$ be an EP.

If there exists a neighborhood $D \subset \mathbb{R}^n$ of $\bar{x}^*$ and an $L: D \rightarrow \mathbb{R}$ s.t. $L(\bar{x}^*) = 0$ and

$$(1) \quad L(\bar{x}) > 0$$

$$(2) \quad \dot{L}(\bar{x}) := \nabla L(\bar{x}) \cdot F(\bar{x}) \leq (<) 0$$

for all $\bar{x} \in D \setminus \{\bar{x}^*\}$, then

L is a (strict) Lyapunov fun for $\bar{x}^*$,
and $\bar{x}^*$ is a stable (asymptotically stable) EP.

$$(ii) \quad \text{Let } L(\bar{x}) = \frac{|\bar{x}|^2}{2} \text{ \& } D = B_1(0).$$

Then $L(0) = 0$ \& (1) holds

$$\text{For (2); } \dot{L}(\bar{x}) = \nabla L \cdot F = \bar{x} \cdot \begin{bmatrix} y \\ -x - \epsilon x^2 y \end{bmatrix}$$

$$= -\varepsilon x^2 y^2 \leq 0 \quad \forall x \in D \setminus \{0\}.$$

$\Rightarrow 0$ is a stable E.P.

(c) Alternative solution: Find L strict Lyapunov.

$$\text{Take } L(x, y) = \frac{|x|^2}{2} - \delta x^3 y$$

$\delta > 0$, $\nabla L(0) = 0$ and Hessian

$$DL(0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \text{ is positive definite}$$

$\Rightarrow L(0)$ is strict local minimum.
 so (1) holds for $D = B_\delta(0)$. $\delta > 0$ small.

$$(2) \dot{L}(x) = -\varepsilon x^2 y^2 + 3\delta x^2 y^2$$

$$- \delta x^4 + \delta \varepsilon x^5 y$$

$$= -(\varepsilon - 3\delta) x^2 y^2$$

$$- \delta x^4 (1 - \varepsilon x y) < 0 \quad \forall x \in B_\delta(0) \setminus \{0\}.$$

if $\delta < \varepsilon_3$ and δ small.

d) Consider
$$\begin{aligned}x' &= y \\ y' &= -\sin x\end{aligned}$$

(i) Explain why this is a Hamiltonian system

(ii) Use the Lyapunov theorem to verify stability of 0. Explain why 0 is not asymptotically stable

(iii) Establish asymptotic stability of
$$\begin{aligned}x' &= y \\ y' &= -\sin x - ky \quad k > 0\end{aligned}$$

Sol (i) Let $H(x, y) = \frac{y^2}{2} - \cos x + 1$

Then
$$\begin{aligned}x' &= H_y \\ y' &= -H_x\end{aligned}$$

(ii) $H(0, 0) = 0$, $\nabla H(0, 0) = 0$ and
$$D^2 H(0, 0) = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \Rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix} \text{ is}$$

a strict local minimum

(L1) holds for a $D \subset \mathbb{R}^2$ small containing 0.

$$(L2) \dot{H}(x, \gamma) = \nabla L \cdot F = \begin{bmatrix} \sin x \\ \gamma \end{bmatrix} \begin{bmatrix} -\gamma \\ -\sin x \end{bmatrix} = 0 \quad \forall (x, \gamma) \in \mathbb{R}^2$$

$\Rightarrow H$ is a Lyapunov fcn for 0.

$\Rightarrow 0$ is stable.

Suppose 0 is A-stable. Then, for (x_0, γ_0) near 0, $\Sigma(t) \rightarrow 0$ as $t \rightarrow \infty$.

$\Rightarrow H(x(t), \gamma(t)) \rightarrow H(0, 0) = 0$, but this contradicts that

$$\frac{d}{dt} H(x(t), \gamma(t)) = 0 \quad \forall t \geq 0,$$

since $H(x_0, \gamma_0) > 0$.

(iii) For this ODE, let

$$L(x) = \frac{y^2}{2} - (\cos x + \delta \sin x) y - 1$$

for $\delta > 0$. Then $L(0) = 0$.

$$(L1) \quad \nabla L(x) = \begin{bmatrix} \sin x + \delta \cos(x) y \\ y + \delta \sin x \end{bmatrix} \text{ satisfies}$$

$$\nabla L(0) = 0 \quad \& \quad D^2 L(0) = \begin{bmatrix} 1 & \delta \\ \delta & 1 \end{bmatrix}$$

is positive definite for $\delta \in (0, 1)$

$\Rightarrow$ 0 is strict local min. & (L1) holds
for $D = B_\delta(0)$ $\delta > 0$ small.

$$(L2) \quad \dot{L}(x) = \nabla L(x) \cdot \begin{bmatrix} y \\ -\sin x - k y \end{bmatrix}$$

$$= (\delta - k) y^2 - \delta \sin^2(x) - \delta k \sin(x) y$$

$$\leq (\delta - k) y^2 - \delta \sin^2(x) + k \left(\frac{y^2}{2} + \delta \frac{\sin^2(x)}{2} \right)$$

$$= \left(\delta - \frac{k}{2} \right) y^2 - \delta \left(1 - \frac{\delta k}{2} \right) \sin^2(x)$$

Take $\delta < \min\left(\frac{\kappa}{2}, \frac{2}{\kappa}\right)$, then

$$\dot{L}(x, y) < 0 \quad \forall (x, y) \in \left(-\frac{\pi}{2}, \frac{\pi}{2}\right) \times \mathbb{R}, \\ (x, y) \neq 0.$$

$\Rightarrow L$ is strict Lyapunov for 0,
so 0 is asymptotically stable.