

Exam 2024 Utseft

Problem 1 consider $x' + x = \sin(t)$

1a Find $\phi_t(x_0)$

sol Integrating factor e^t :

$$(e^t x)' = e^t \sin t \quad \int$$

Integration by parts:

$$\begin{aligned} \int e^t \sin t \, dt &= e^t \sin t + C - \int e^t \cos t \, dt \\ &= e^t (\sin t - \cos t) + C - \int e^t \sin t \, dt \end{aligned}$$

$$\Rightarrow \int e^t \sin t \, dt = \frac{1}{2} e^t (\sin t - \cos t) + C$$

$$\Rightarrow x(t) = e^{-t} (x_0 + C) + \frac{1}{2} (\sin t - \cos t)$$

$$x(0) = x_0 \Rightarrow C = \frac{1}{2} \quad \text{so}$$

$$\phi_t(x_0) = e^{-t} \left(x_0 + \frac{1}{2} \right) + \frac{1}{2} (\sin t - \cos t)$$

16 Let $x_{n+1} = \phi_{2\pi}(x_n)$ with x_0 given
Show that sequence converges and
compute $\lim_{n \rightarrow \infty} x_n$.

Sol $\phi_{2\pi}(x_n) = e^{-2\pi} \left(x_n + \frac{1}{2}\right) - \frac{1}{2}$

$$\begin{aligned} \text{So } |x_{n+1} - x_n| &= e^{-2\pi} |x_n - x_{n-1}| \\ &= e^{-2n\pi} |x_1 - x_0| \end{aligned}$$

Hence for $n > m$

$$\begin{aligned} |x_n - x_m| &\leq \sum_{k=m}^{n-1} |x_{k+1} - x_k| \\ &\leq e^{-2m\pi} |x_1 - x_0| \underbrace{\sum_{k=0}^{\infty} e^{-2k\pi}}_{\leq 2} \\ &\rightarrow 0 \text{ as } m, n \rightarrow \infty \end{aligned}$$

$\Rightarrow \{x_m\}$ is Cauchy in $(\mathbb{R}, |\cdot|)$.

$\Rightarrow \lim_{m \rightarrow \infty} x_m =: \bar{x}$ exists.

To determine limit:

$$\bar{x} = \lim_{n \rightarrow \infty} x_{n+1} = \lim_{n \rightarrow \infty} \phi(x_n) \stackrel{\phi \text{ continuous}}{=} \phi_{2\pi}(\bar{x})$$

$$\Rightarrow \bar{x} \text{ solves } \bar{x} = (\bar{x} + \frac{1}{2})e^{-2\pi} - \frac{1}{2}$$

$$\Rightarrow \bar{x}(1 - e^{-2\pi}) = \frac{1}{2}(e^{-2\pi} - 1)$$

$$\Rightarrow \bar{x} = -\frac{1}{2}.$$

1c A 2π periodic solution must satisfy

$\phi_{2\pi}(x_0) = x_0$. Part b shows that only $x_0 = -\frac{1}{2}$ satisfies this property.

Further, $\phi_t(-\frac{1}{2}) \stackrel{1a}{=} \frac{1}{2}(\sin t - \cos t)$
clearly is 2π periodic.

Problem 2 Consider the system

$$x' = x - y - x^3$$

$$y' = x + y - y^3$$

2a Find all EP and classify them.

Sol 0 is an EP. suppose $x \neq 0$

is an EP. Then

$$y = x - x^3$$

$$x = y^3 - y$$

imply that $x \neq 0 \Leftrightarrow y \neq 0$.

$$\text{and } y = y^3 - y - (y^3 - y)^3$$

$$\Rightarrow 1 = \underbrace{(y^2 - 1)(1 - y^2(y^2 - 1))}_{g(y^2)}$$

But this has no solution $y^2 \neq 0$
as $|g(y^2)| < 1$ for $y^2 < 2$

and $g(y^2) < 0$ for $y^2 \geq 2$.

$$DF(0) = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_A$$

Char eq $\lambda^2 - 2\lambda + 2 = 0$

$\lambda_{\pm} = -1 \pm i$, So A is hyperbolic

and Linearization thm $\Rightarrow 0$ is a spiral sink

2b show that for $x = r \cos \theta$, $y = r \sin \theta$, it holds that

$$\frac{1}{2} \frac{d}{dt} r^2 = r^2(1-r^2) + \frac{1}{2} r^2 \sin^2(2\theta)$$

$\uparrow$ typo: should be r^4

Sol

$$\frac{1}{2} \frac{d}{dt} r^2 = \frac{1}{2} \frac{d}{dt} |\mathbf{x}|^2 = \mathbf{x} \cdot \mathbf{F}(\mathbf{x})$$

$$= x^2 + y^2 - (x^4 + y^4)$$

$$= x^2 + y^2 - (x^2 + y^2)^2 + 2x^2y^2$$

$$= r^2(1-r^2) + 2r^4(\cos\theta \sin\theta)^2$$

$$= r^2(1-r^2) + \frac{1}{2} r^4 \sin^2(2\theta)$$

2c Show that for any $\varepsilon \in (0, 1)$

$$\left. \frac{d}{dt} r^2 \right|_{r=\sqrt{1-\varepsilon}} > 0 \quad \text{and} \quad \left. \frac{d}{dt} r^2 \right|_{r=\sqrt{2+\varepsilon}} < 0.$$

Sol

For $r^2 = 1 - \varepsilon$

$$\left. \frac{d}{dt} r^2 \right|_{r^2=1-\varepsilon} = \underbrace{2\varepsilon(1-\varepsilon)}_{>0} + \underbrace{(1-\varepsilon)^2 \sin^2(2\theta)}_{\geq 0} > 0$$

for any $\theta \in [0, 2\pi)$.

For $r^2 = 2 + \varepsilon$

$$\begin{aligned} \left. \frac{d}{dt} r^2 \right|_{r^2=2+\varepsilon} &= -2(2+\varepsilon)(1+\varepsilon) + (2+\varepsilon)^2 \underbrace{\sin^2(2\theta)}_{\leq 1} \\ &\leq (2+\varepsilon)(2+\varepsilon - 2 - 2\varepsilon) \\ &= -\varepsilon(2+\varepsilon) < 0. \end{aligned}$$

2d Explain why this implies that

$D = \{r \mid 1 \leq r \leq \sqrt{2}\}$ has a closed orbit.

Sol By continuity $\left. \frac{d}{dt} r^2 \right|_{r=1} = \lim_{\varepsilon \rightarrow 0} \left. \frac{d}{dt} r^2 \right|_{r=\sqrt{1-\varepsilon}} \geq 0$

and similarly $\frac{d}{dt} r^2 \Big|_{r=\sqrt{2}} \leq 0$.

Hence D is positively invariant, compact, has no EP, so Poincaré-Bendixson thm $\Rightarrow$ it has at least one closed orbit.

Problem 3 For ODE $x' = f(x)$, recall

$$x_{n+1} = x_n + h f(x_n) \quad (\text{forward Euler})$$
$$x_{n+1} = x_n + h f(x_{n+1}) \quad (\text{implicit Euler})$$

Consider IVP

$$\begin{aligned} x' &= x^{1/3} & t > 0 \\ x(0) &= 0 \end{aligned} \quad (3)$$

3a Find all solutions of (3).

Sol: Any solution must satisfy $x(t) \geq 0$.

Suppose $x(\bar{t}) > 0$ for a solution and some $\bar{t} > 0$. Then, locally, it must satisfy

$$\underbrace{\int x^{-1/2} dx}_{2x^{1/2}} = t - c$$

$$\Rightarrow X(t) = \frac{1}{4}(t-c)^2 \quad \text{with } c = 2\sqrt{X(0)} + t$$

This solution extends uniquely to

$$X(t) = \frac{1}{4}(t-c)^2 \quad t \in (c, \infty)$$

and as $x' \geq 0$, also uniquely further to

$$X_c(t) = \begin{cases} 0 & t \in [0, c] \\ \frac{1}{4}(t-c)^2 & t > c \end{cases}$$

for any $c > 0$.

3b Show that forward Euler fails to approximate the solution with $X(1) = \frac{1}{4}$ (i.e. $X_{c=0}(t)$)

$$\text{Sol } X_{n+1} = X_n + h f(X_n) \Rightarrow$$

$$X_1 = 0 + h f(0) = 0$$

By iteration

$$x_n = 0 \quad \forall n. \quad \text{Hence}$$

$$x_{n-1} = 0 \neq \frac{1}{4}.$$

3c Let $\{x_n\}_{n \geq 0}$ be generated by implicit Euler of (3) and make ansatz that it is on the form

$x_n = (c_n n h)^2$. Show that $c_0 = 0$ and $c_1 = 1$ is a solution, and find formula for c_n s.t. $c_{n+1} = F(n, c_n)$.

So
$$x_1 = x_0 + h f(x_1) = h \sqrt{x_1}$$

$$\begin{aligned} \text{Eq: } (c_1 h)^2 &= h c_1 h \Leftrightarrow c_1^2 = c_1 \\ &\Leftrightarrow c_1 \in \{0, 1\} \end{aligned}$$

Ok. Formula:

$$x_{n+1} = x_n + h f(x_{n+1})$$

$$(c_{n+1} (n+1) h)^2 = (c_n n h)^2 + h^2 c_{n+1} (n+1).$$

