

Exam 2025

Consider $x'(t) = P(t)x(t)$ $t \in \mathbb{R}$,
where $P: \mathbb{R} \rightarrow \mathbb{R}$ is T -periodic and
satisfies $\int_0^T P(t) dt = 0$.

Show that $x(t)$ is T -periodic.

Sol Integrating factor $e^{-\int_0^t P(s) ds}$

$$\Rightarrow \frac{d}{dt} \left(x(t) e^{-\int_0^t P(s) ds} \right) = 0$$

$$\Rightarrow x(t) = x_0 e^{\int_0^t P(s) ds}$$

is unique solution. Further

$$\begin{aligned} x(t+T) &= x_0 e^{\int_0^{t+T} P(s) ds} \\ &= x_0 e^{\int_0^T P(s) ds} \cdot e^{\int_T^{T+t} P(s) ds} \\ &= x_0 \cdot e^0 \cdot e^{\int_0^t P(s+T) ds} \quad \left(\begin{array}{l} P(s+T) \\ = P(s) \end{array} \right) \\ &= x_0 e^{\int_0^t P(s) ds} = x(t). \end{aligned}$$

Problem 2 Consider

$$\dot{X} = \underbrace{\begin{bmatrix} 0 & -1 \\ 1 & \alpha \end{bmatrix}}_{A(\alpha)} X \quad \alpha \in \mathbb{R}.$$

Discuss nature of phase portrait for different values of α .

sol Char eq $\lambda^2 - \alpha\lambda + 1 = 0$

$$\lambda_{\pm} = \frac{\alpha \pm \sqrt{\alpha^2 - 4}}{2}$$

$\alpha \geq 2$: source $\lambda_{\pm} > 0$

$\alpha \in (0, 2)$: spiral source $\operatorname{Re}(\lambda_{\pm}) > 0, \operatorname{Im}(\lambda_{\pm}) \neq 0$

$\alpha = 0$: center $\operatorname{Re}(\lambda_{\pm}) = 0, \operatorname{Im}(\lambda_{\pm}) = \pm i$

$\alpha \in (-2, 0)$: spiral sink $\operatorname{Re}(\lambda_{\pm}) < 0, \operatorname{Im}(\lambda_{\pm}) \neq 0$

$\alpha \leq -2$: sink $\lambda_{\pm} < 0$.

Problem 3 Consider

$$x' = ax - x^5 \quad a \in \mathbb{R}.$$

Find bifurcation point and analyse nature, number and location of EPs.

Sol

$$f_a(x) = x(a - x^4) = 0$$

$$\text{EP: } x_0 = 0$$

$$a \leq 0$$

$$x_0 = 0, x_{\pm} = a^{1/4}$$

$$a > 0$$

Bifurcation point: $a = 0$

$$f'_a(x) = a - 5x^4$$

$\Rightarrow f'_a(x_0) = a$, so by Linearization,

x_0 is a $\begin{cases} \text{sink} & \text{for } a < 0 \\ \text{A-stable} & \text{for } a = 0 \\ \text{source} & \text{for } a > 0 \end{cases}$

$a = 0$ Phase line $\Rightarrow$ A-stable



$$f'_a(x_{\pm}) = a - 5a = -4a < 0$$

$\Rightarrow x_{\pm}(a)$ are smks for $a > 0$.

Problem 4 Consider $x' = ax$ for $a < 0$.

4a Write down numerical methods forward Euler, implicit Euler and Crank-Nicolson for a given $h > 0$.

Sol Forward Euler

$$x_{n+1} = x_n + h a x_n = x_n(1 + ah)$$

Implicit Euler

$$x_{n+1} = x_n + h a x_{n+1} \Rightarrow x_{n+1} = \frac{x_n}{1 - ah}$$

Crank-Nicolson

$$x_{n+1} = x_n + \frac{h}{2} (a x_n + a x_{n+1})$$

$$\Rightarrow x_{n+1} = \frac{1 + ah/2}{1 - ah/2} x_n$$

4b For all three methods, discuss how large $h > 0$ can be to ensure that $|x_n|$ remains bounded for any n and any x_0 .

Sol Forward Euler

$$|x_n| = |1 + ah|^n |x_0|$$

$$\text{bounded} \Leftrightarrow |1 + ah| \leq 1 \Leftrightarrow 1 + 2ah + a^2h^2 \leq 1$$

$$\Leftrightarrow h \leq -\frac{2}{a}$$

Implicit Euler

$$|x_n| = |1 - ah|^{-n} |x_0|$$

$$\text{bounded} \Leftrightarrow 1 \leq |1 - ah| = 1 - ah$$

which is true for all $h > 0$ as $a < 0$.

Crank-Nicolson

$$|x_n| = |1 - ah/2|^{-n} |1 + ah/2| |x_0|$$

$$\text{bounded} \Leftrightarrow |1 + ah/2|^2 \leq |1 - ah/2|^2$$

$$\Leftrightarrow ah \leq 0$$

holds for any $h \geq 0$.

Problem 5 $x' = -V'(x)$, $V(x) = (x^2 - 1)^2$

a) $x' = -4x(x^2 - 1) =: f(x)$

EP: $x_0 = 0$, $x_{\pm} = \pm 1$

$$f'(x) = -12x^2 + 4 \quad \text{and}$$

$$f'(x_0) = 4, \quad f'(x_{\pm}) = -8.$$

So by linearization then

x_0 is a source (unstable)

$x_{\pm}$ are sinks (A-stable).

b) We consider

$$\dot{x} = -\nabla V(x) \quad \text{where}$$

$$V(x) = (x^2 + y^2 - 1)^2$$

Equilibria of

$$\dot{x} = -\nabla V = - \begin{bmatrix} 4x(x^2 + y^2 - 1) \\ 4y(x^2 + y^2 - 1) \end{bmatrix} = F(x)$$

Any $(x, y) \in \mathbb{R}^2$ s.t. $|x| = 1$

(i.e. all $x \in \partial B_1(0)$).

And $(x, y) = 0$.

Stability of equilibria:

$$DF(x) = - \begin{bmatrix} 12x^2 + 4y^2 - 4 & 8xy \\ 8xy & 12y^2 + 4x^2 - 4 \end{bmatrix}$$

$DF(0) = \begin{bmatrix} 4 & 0 \\ 0 & 4 \end{bmatrix}$ is hyperbolic.
Linearization then $\Rightarrow 0$ is unstable

For $|\mathbf{x}|=1$,

$$DF(\mathbf{x}) = - \begin{bmatrix} 8x^2 & 8xy \\ 8xy & 8y^2 \end{bmatrix}$$

Char eq. $\lambda^2 + 8\lambda = 0$

$$\lambda_1 = 0, \lambda_2 = -8.$$

Cannot apply linearization thm.

Polar coordinates $x = r \cos \theta, y = r \sin \theta$

$$\Rightarrow \frac{1}{2} \frac{dr^2}{dt} = \mathbf{x} \cdot F(\mathbf{x}) = -4r^2(r^2 - 1)$$

$$\Rightarrow r' = 8r(1 - r^2)$$

To compute θ' , note that for $x \neq 0$, $\tan \theta = \frac{y}{x}$

$$\Rightarrow \theta' = \frac{d}{dt} \arctan\left(\frac{y}{x}\right) = \frac{1}{1 + \left(\frac{y}{x}\right)^2} \cdot \underbrace{\left(\frac{y}{x}\right)'}_{=0} = 0 \quad \text{And for}$$

$$y \neq 0, \cot \theta = \frac{x}{y} \Rightarrow \theta' = -\frac{1}{1 + \left(\frac{x}{y}\right)^2} \underbrace{\left(\frac{x}{y}\right)'}_{=0} = 0$$

Hence $r' = 8r(1-r^2)$

$$\theta' = 0$$

For any point $\mathbb{X}^*$ on $\partial B_1(0)$, in polar coordinates $\mathbb{X}^* = (1, \theta^*)$, the stability of

$$r' = 8r(1-r^2) = f_r(r)$$

(by linearization $f_r'(r) = 8 - 24r^2$,
so $f_r'(1) = -16$)

implies that for any small $\delta > 0$ and $\mathbb{X}_0$ near $\mathbb{X}^*$: in polar coordinates
for any $(r_0, \theta_0) \in (1-\delta, 1+\delta) \times (\theta^*-\delta, \theta^*+\delta)$,
it holds that $\lim_{t \rightarrow \infty} r(t) = 1$, $\theta(t) = \theta_0 \forall t$.

Hence $(r(t), \theta(t)) \in (1-\delta, 1+\delta, \theta^*-\delta, \theta^*+\delta)$

for all $t \geq 0$, implying that any $\mathbb{X}^* \in \partial B_1(0)$ is stable.

But choosing $(r_0, \theta_0) = (1, \theta_0)$ with θ_0 arbitrarily close to θ^* ,

shows that $\lim_{t \rightarrow \infty} (r(t), \theta(t)) = (1, \theta_0) \neq (1, \theta^*)$

so all equilibria $(1, \theta^*)$ $\theta^* \in [0, 2\pi)$ are stable but not A-stable.

Problem 6

$$\left. \begin{aligned} x' &= -L_y(x, y) M(x, y) \\ y' &= L_x(x, y) M(x, y) \end{aligned} \right\} =: F(x, y) \quad (6)$$

where $M(x, y) = 0$ if $x=0$ or $y=0$.

a) For a solution $(x(t), y(t))$,

$$\begin{aligned} \frac{dL(x(t), y(t))}{dt} &= L_x x' + L_y y' \\ &= (-L_x L_y + L_y L_x) M = 0. \end{aligned}$$

b) $x=0$ is an x -nullcline
and $y=0$ is an y -nullcline

this shows that if $(x_0, y_0) \in (0, \infty)^2$,
then $x(t) \geq 0, y(t) \geq 0$ for all $t \geq 0$ for
which the unique solution is defined*

(*The constraints on L and M do not
suffice to ensure that $\Phi_t(x_0)$ is
well-defined $\forall t \geq 0$. We imposed this
as a condition for positive invariance
in the lectures this year.)

To sharpen this to $x(t) > 0$ & $y(t) > 0$,
suppose $\exists \bar{t} > 0$ s.t. $x(\bar{t}) = 0$. But
then $M(x(\bar{t}), y(\bar{t})) = 0$ and
the unique solution of the ODE
(6) with initial condition
 $(x(\bar{t}), y(\bar{t})) = (0, y(\bar{t}))$ becomes
 $(x(t), y(t)) = (0, y(\bar{t})) \quad \forall t \in \mathbb{R}$,
contradicting $x_0 > 0$.

c) $L: \mathbb{R}^2 \rightarrow \mathbb{R}$ has a strict minimum
in $(x_0, y_0) \Rightarrow \nabla L(x_0, y_0) = 0$

$\Rightarrow L_x(x_0, y_0) = L_y(x_0, y_0) = 0$. Hence

$$F(\underline{x}_0) = \begin{bmatrix} -L_y(x_0, y_0) & M(x_0, y_0) \\ L_x(x_0, y_0) & M(x_0, y_0) \end{bmatrix} = 0,$$

so $\underline{x}_0$ is an EP. For

$$\tilde{L}(\underline{x}) := L(\underline{x}) - L(\underline{x}_0),$$

there exists some neighborhood
 D of $\underline{x}_0$ s.t.

$$\tilde{L}(\underline{x}) > \tilde{L}(\underline{x}_0) = 0 \quad \forall \underline{x} \in D \setminus \{\underline{x}_0\}$$

and $\dot{\tilde{L}}(\underline{x}) = 0 \quad \forall \underline{x}$ by part a.

$\Rightarrow \tilde{L}$ is a Lyapunov fun for $\underline{x}_0$
so $\underline{x}_0$ is stable.

Suppose $\bar{X}_0$ is asymptotically stable. Then for $\tilde{X}_0 \neq \bar{X}_0$ sufficiently close to $\bar{X}_0$, it holds that $\boxed{1}$ $L(\tilde{X}_0) > L(\bar{X}_0)$ and $\boxed{2}$ $\lim_{t \rightarrow \infty} X(t; \tilde{X}_0) = \bar{X}_0$

But by part a) $L(X(t; \tilde{X}_0)) = L(\tilde{X}_0) \forall t$, contradicting $\boxed{1}$ as

$$\begin{aligned} L(\bar{X}_0) &\stackrel{\boxed{2}}{=} L\left(\lim_{t \rightarrow \infty} X(t; \tilde{X}_0)\right) \\ &= \lim_{t \rightarrow \infty} L(X(t; \tilde{X}_0)) \\ &\stackrel{a)}{=} L(\tilde{X}_0). \end{aligned}$$