

Lec 1

19/1

MAT 3440

Syllabus: Differential eq ... chaos 3rd ed
by Hirsch, Smale & Devaney

Most of § 1-6 (Linear ODE)

Parts of § 7-14, 17 (Nonlinear ODE)

Note on Numerical methods for ODEs
by U. Fjordholm.

Contents of course

- Extensive theory for linear ODEs

$$X'(t) = Ax$$

- Existence & uniqueness for some
nonlinear ODEs $X' = f(x)$

- Qualitative studies of nonlinear

ODEs, e.g. dynamics near points $x^* \in \mathbb{R}^d$ where $f(x^*) = 0$,

closed-loop solutions
orbits



- Applications in biology (SIR and Lotka-Volterra), electronics, mechanics and chaos theory (Lorenz systems).
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Today: ODEs in 1D

- Simplest example $x' = ax$ (Existence & uniqueness)
- Classification of different ODEs.
- Equilibrium points and bifurcations

Simplest ODE

For $a \in \mathbb{R}$ and non-empty interval $I \subset \mathbb{R}$, we seek $X: I \rightarrow \mathbb{R}$ s.t.

$$X'(t) = a X(t) \quad \forall t \in I \quad (1)$$

compactly written $X' = aX$.

Interpretation:

$$\underbrace{X'(t)}_{\text{rate of change}} = a X(t) \quad \text{const} \times \text{current state}$$

Applications: Bank account (cont. time)
 $a = \text{interest rate}$

Population growth (typically with $a > 0$)

Isotope decay, (e.g. Carbon dating)
of organism.

$X(t) = {}^{14}\text{C}$ concentration in organism.

$$X'(t) = \underbrace{-0.000121}_{=:a} \cdot X(t)$$

Time of death: $t=0$, and $X(0)$ often known.

Half-life of ${}^{14}\text{C}$, T s.t. $\frac{X(T)}{X(0)} = \frac{1}{2}$,
is 5730 years.

Dating dead organism:

Let y = measured ${}^{14}\text{C}$ concentration

Time since death: $t \approx \frac{\ln\left(\frac{y}{X(0)}\right)}{a}$

(as $X(t) = X(0)e^{at} \approx y$).

Solving eq. (1) for $I = \mathbb{R}$ & $a \neq 0$

$$X'(t) = aX(t) \quad | \cdot e^{-at} \leftarrow \begin{array}{l} \text{Integrating} \\ \text{factor} \end{array}$$

$$\Rightarrow X'(t)e^{-at} - aX(t)e^{-at} = 0$$

$$\Rightarrow \frac{d}{dt} (x e^{-at}) = 0 \quad | \quad \int_0^t ds$$

$$\Rightarrow x(t) = x_0 e^{at}$$

Given $x(0) = x_0$, the solution is unique
(because integration $\int_0^t \frac{d}{ds} (x(s) e^{-as}) ds$
yields unique value).

Alternative argument:

Suppose $y(t)$ also solves $\begin{matrix} x' = at \\ x(0) = x_0 \end{matrix}$

Then

$$\begin{aligned} \frac{d}{dt} (y e^{-at}) &= y' e^{-at} - a y e^{-at} \\ &= 0 \end{aligned}$$

$$\Rightarrow y e^{-at} = c \Rightarrow y(t) = c e^{at}$$

$$\text{and } y(0) = x_0 \Rightarrow y(t) = x_0 e^{at}$$

Classification of ODEs

$x'(t) = f(x(t), t)$ with $f: \mathbb{R}^d \times I \rightarrow \mathbb{R}^d$
is called

- First-order, since derivative is of order 1. ($x'' = f(x', x, t)$ would be second order etc.)
- Autonomous if $f = f(x)$

Ex $x' = x^2$

Non-autonomous if $f = f(x, t)$

Ex $x' = x^2 + \sin(t)$.

- If $x' = a(t)x + g(t)$ (2)

for some $a: I \rightarrow \mathbb{R}^{d \times d}$ and $g: I \rightarrow \mathbb{R}^d$,

then the ODE is linear.

- For linear ODE (2), if $g \equiv 0$, then it is homogeneous.

Otherwise, it is inhomogeneous.

- A (first-order) ODE not on the form (2) is called nonlinear

Ex $x' = x^2$

Problem Solve $\left. \begin{array}{l} x' = ax + b \\ x(0) = x_0 \end{array} \right\} (3)$

where $a, b \in \mathbb{R}$ with $a \neq 0$.

Multiply with e^{-at} :

$$\underbrace{(x' - ax)}_{\frac{d}{dt}(xe^{-at})} e^{-at} = e^{-at} b$$

$$\frac{d}{dt}(xe^{-at})$$

and integrate $\int_0^t dt$

$$X(t) = e^{at} x_0 + (e^{at} - 1) \frac{b}{a}$$

Verification: $X'(t) = a e^{at} x_0 + a e^{at} \frac{b}{a}$
 $= \dots = a X(t) + b.$

Uniqueness: Suppose $Y(t)$ also solves (3).

Then $Z(t) := X(t) - Y(t)$ satisfies

$$Z' = X' - Y' = a(X' - Y') = aZ'$$

$$Z(0) = X(0) - Y(0) = 0.$$

From earlier $\Rightarrow Z = Z_0 e^{at} = 0$ is
unique solution $\Rightarrow Y = X.$

Definition

For $x' = f(x)$ any point x^* s.t. $f(x^*) = 0$ is called an equilibrium point (EP)

Importance: If $x_0 = x^*$, then $x(t) = x^*$ is a solution.

If any solution of $\begin{matrix} x' = f(x) \\ x(0) = x_0 \end{matrix}$ with x_0 near x^* satisfies

$\lim_{t \rightarrow \infty} x(t) = x^*$, then x^*

is called A-stable (Asymptotically stable).

(It is called stable if solutions with x_0 near x^* remain near x^* for all times.)

It is called unstable if it is not stable.

Ex Consider again

$$x' = ax =: f_a(x) \quad (4)$$

with $a \in \mathbb{R}$. (This is a family of ODE parametrized in a .)

EP: If $a = 0$ then all $x^* \in \mathbb{R}$
(these points are all stable but not A-stable)

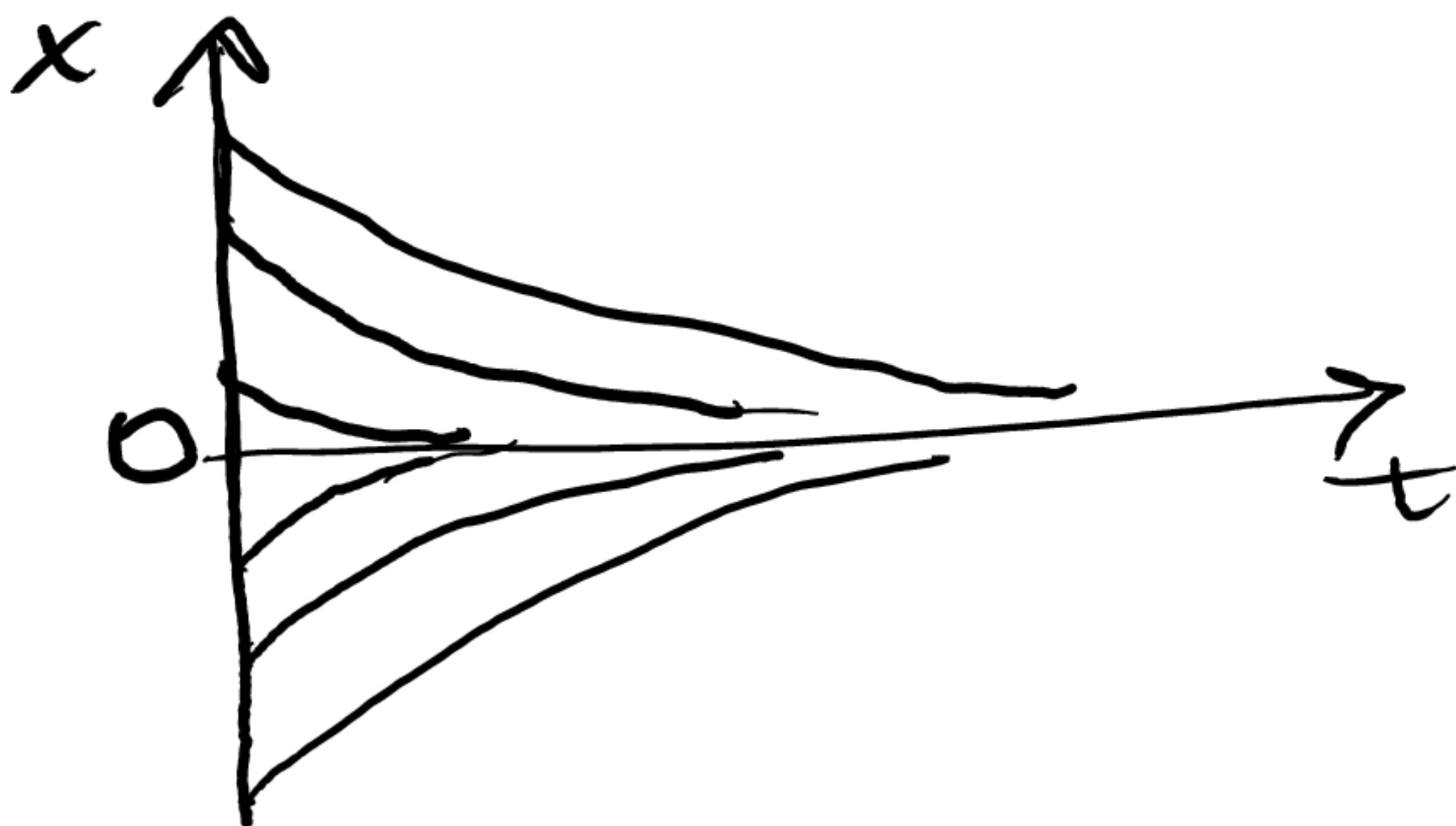
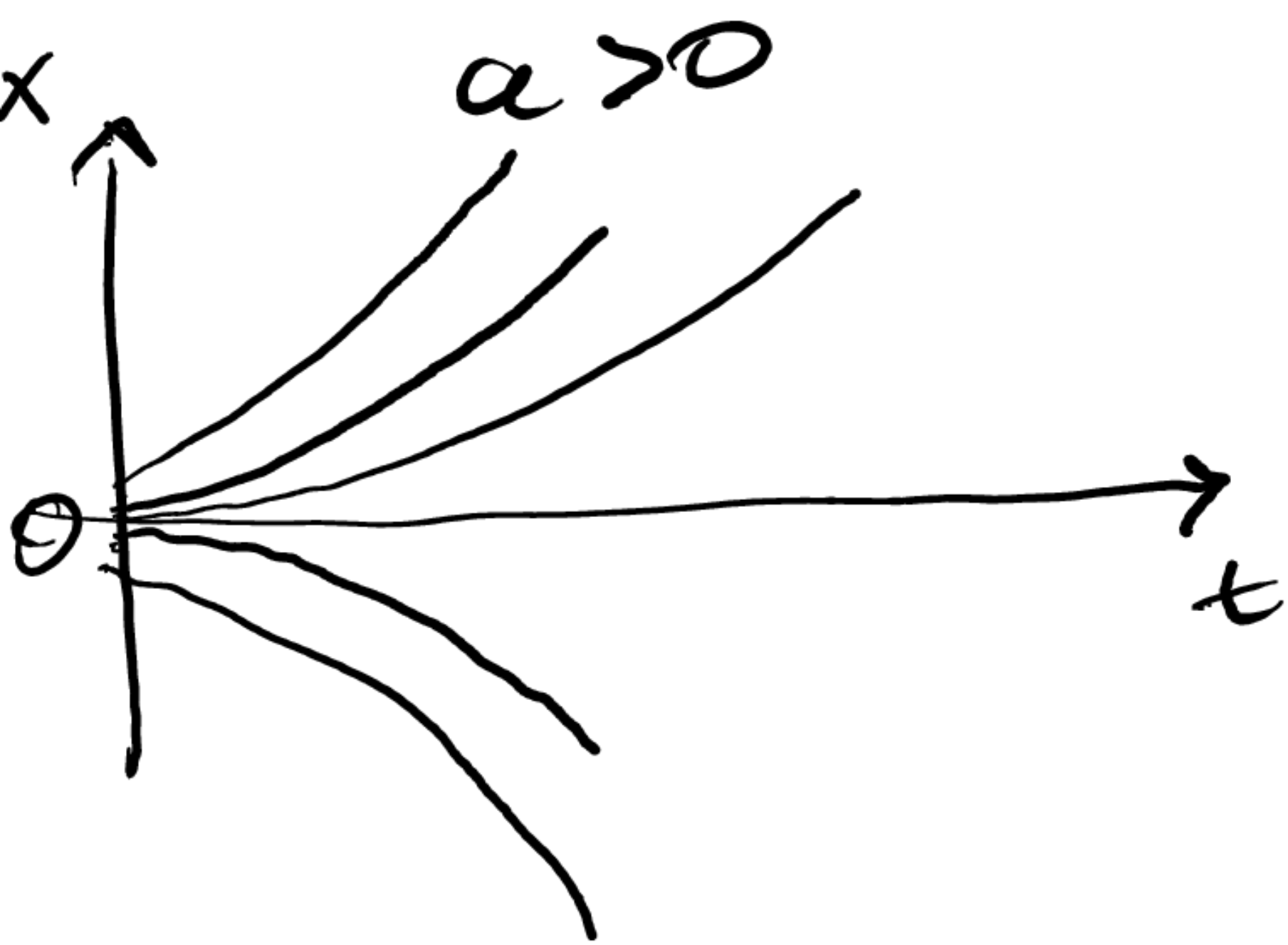
If $a \neq 0$ then only $x^* = 0$.

Stability As $x(t) = x_0 e^{at}$

$$\lim_{t \rightarrow \infty} x(t) = 0 \quad \forall x_0 \text{ near } 0 \quad \Leftrightarrow a < 0,$$

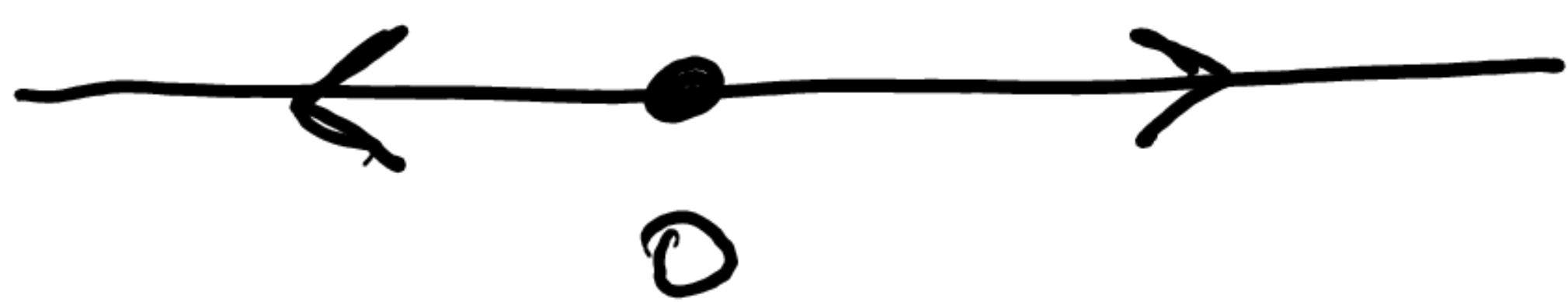
So $x^* = 0$ is A-stable $\Leftrightarrow a < 0$.

$$a < 0$$



Phase line of EP

$$a > 0$$



$$a < 0$$



An A-stable EP is called a sink if $f'(x^*) < 0$.

There exists a $\delta > 0$ s.t.

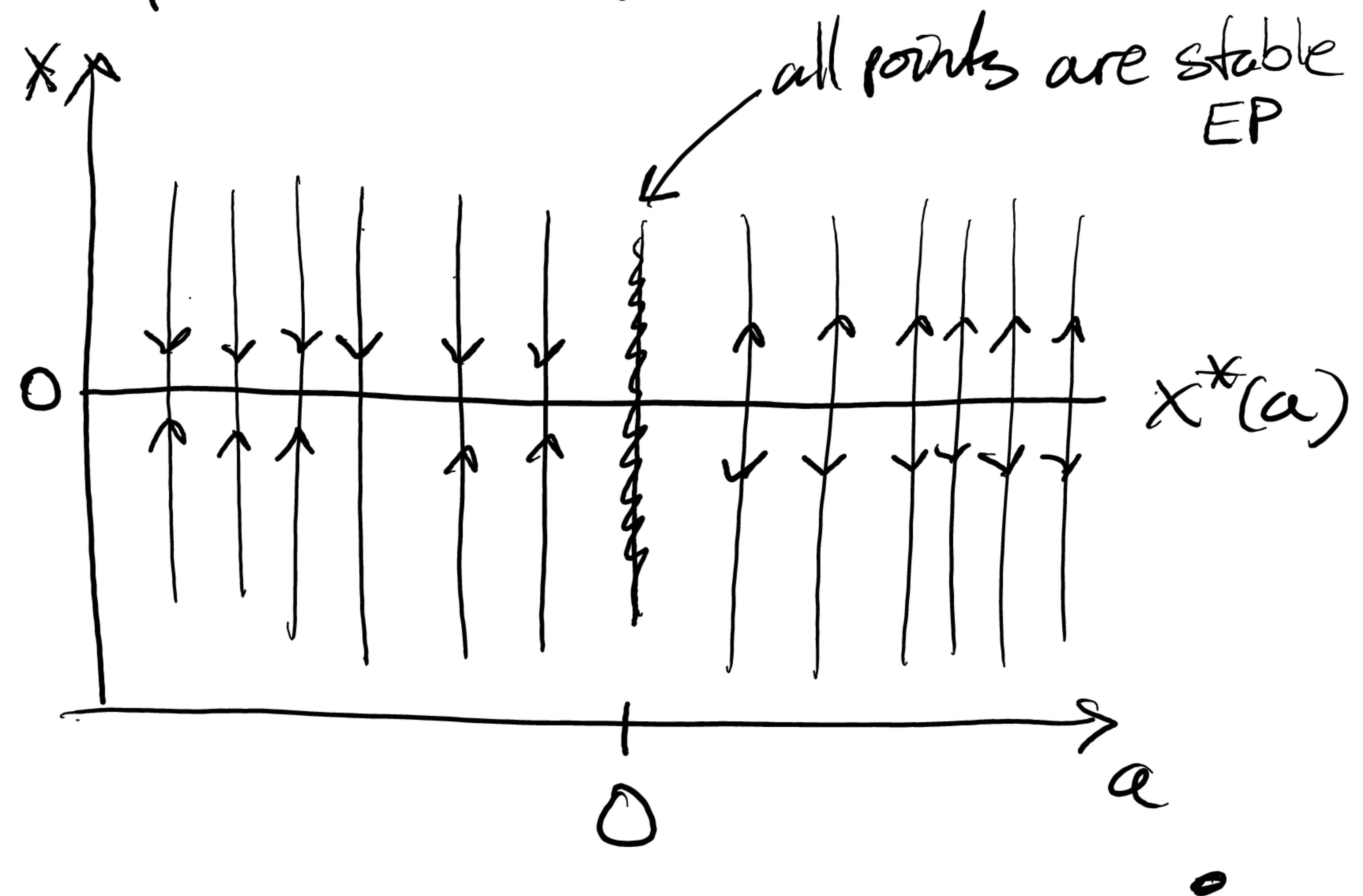
$$X(t; x_0) \rightarrow x^* \quad \forall x_0 \in B(x^*, \delta).$$

Oppositely, if $X(t; x_0)$ move away from x^* for all $x_0 \neq x^*$ near x^* ,

then x^* is a source.

Observation The type of EP $x^* = 0$ for family of ODE (4) changes at $a = 0$; it is called a bifurcation point (a split between A-stable $a < 0$ and unstable $a > 0$).

Bifurcation diagram



Next time Nonlinear population models.