

Lecture 2, 21/1

Previously, we looked at

$$x' = ax =: f_a(x) \text{ for } a \in \mathbb{R},$$

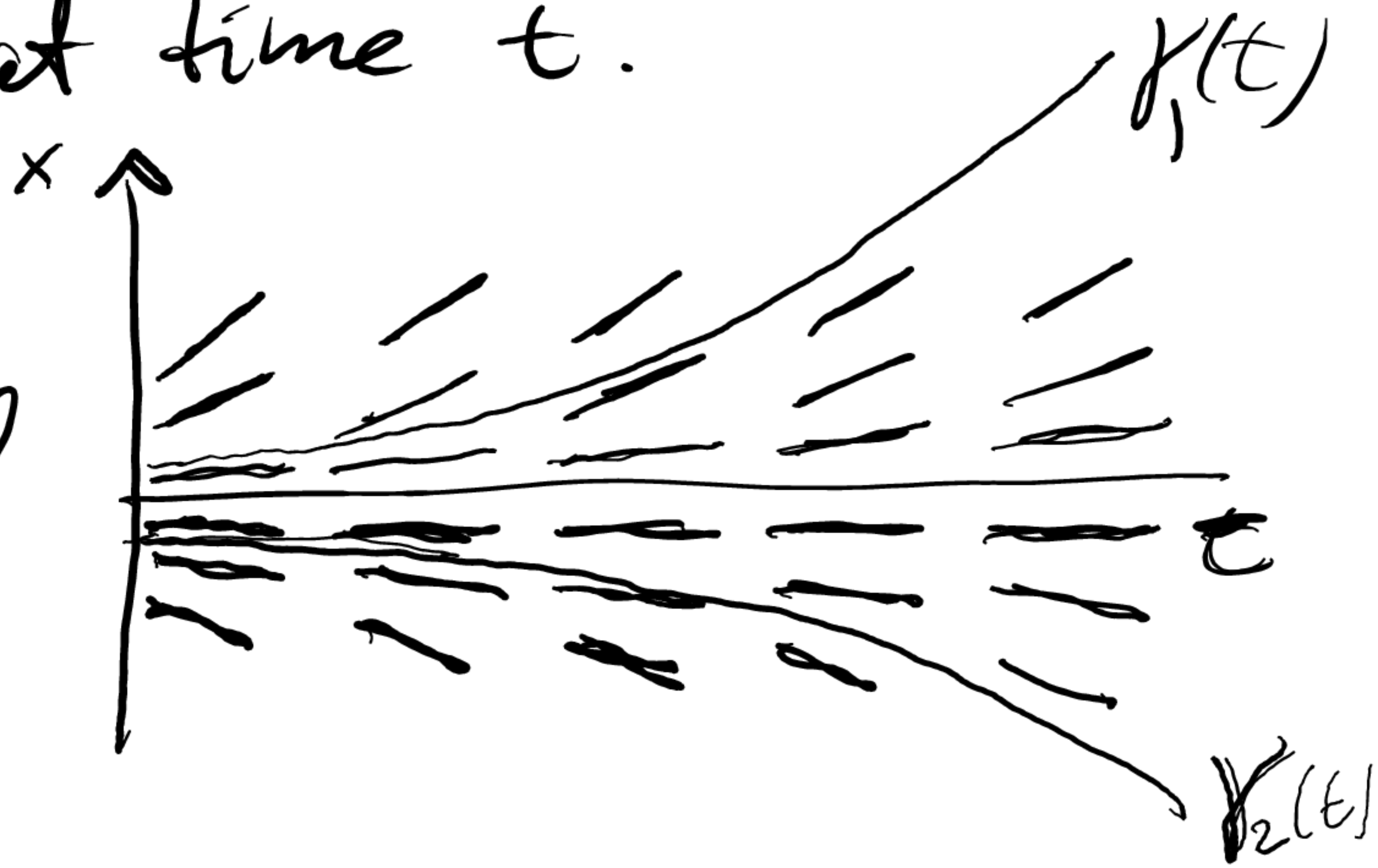
with solutions $x(t) = ce^{at}$

The curve $\gamma(t) = (t, x(t))$ is a solution graph.

Its tangent $\gamma'(t) = (1, x') = (1, ax)$ is the slope at (t, x) of a sol curve.

For $x \in \mathbb{R}$, $(1, f_a(x)) = \text{slope}$ at (t, x) of solution curve going through x at time t .

Slope field
for setting with
 $a > 0$.



Plan

- Separation of variables
 - Population models
 - Exercises
-

Separation of Variables

Consider 1 autonomous ODE

$$x'(t) = f(x(t)) \quad t \in I \quad (1)$$

Suppose/look for a solution s.t.

$$f(x(t)) \neq 0 \quad \forall t \in I. \text{ Then}$$

$$\frac{x'(t)}{f(x(t))} = 1$$

and $x(t)$ is invertible on I
(monotone increasing or decreasing).

$$\Rightarrow \int_{t_0}^t \frac{1}{f(x(s))} x'(s) ds = \int_{t_0}^t 1 ds = t - t_0$$

$$(t_0, t) \subset I$$

Change of variables $u = x(s)$

$$\Rightarrow \frac{du}{ds} = x'(s) = f(x(s)) = f(u)$$

and

$$\int_{x(t_0)}^{x(t)} \frac{1}{f(u)} du = t - t_0$$

If $F'(u) = \frac{1}{f(u)}$ holds for $u \in X(I)$,

then

$$F(x(t)) - F(x(t_0)) = t - t_0$$

$$\Rightarrow x(t) = F^{-1}(t + C) \quad \text{when}$$

F is invertible.

General solution (under same assumption $f \neq 0$ over implicit interval)

$$\int \frac{1}{f(x)} dx = \int 1 dt$$

$$\Rightarrow F(x) = t + C$$

$$\Rightarrow x(t) = F^{-1}(t + C)$$

Ex

$$x' = x^3$$

$$\text{Sol: } \int \frac{1}{x^3} dx = \int 1 dt$$

$$-\frac{x^{-2}}{2} = t + C$$

$$x(t) = \frac{1}{\sqrt{\hat{C} - 2t}}$$

(could have
+/-)

(solution for $t \in (-\infty, \hat{C}/2)$.)

Population models

$x' = ax$ models exponential growth of population, as
 $x(t) = x_0 e^{at}$.

Not accurate over longer time scales,
as $\lim_{t \rightarrow \infty} x(t) = \infty$ if $a > 0$.

Logistic population model

$x' = ax(1 - \frac{x}{N}) =: f_a(x)$ (Nonlinear & autonomous)
where $a > 0$ population growth rate
and $N > 0$ "ideal" pop. size.

Observe: For $x > 0$,

$$f_a(x) \approx \begin{cases} ax & \text{when } x \ll N \\ \text{small} & \text{ } \\ \text{negative} & \text{ } \end{cases} \begin{cases} & x \text{ near } N \\ & x > N. \end{cases}$$

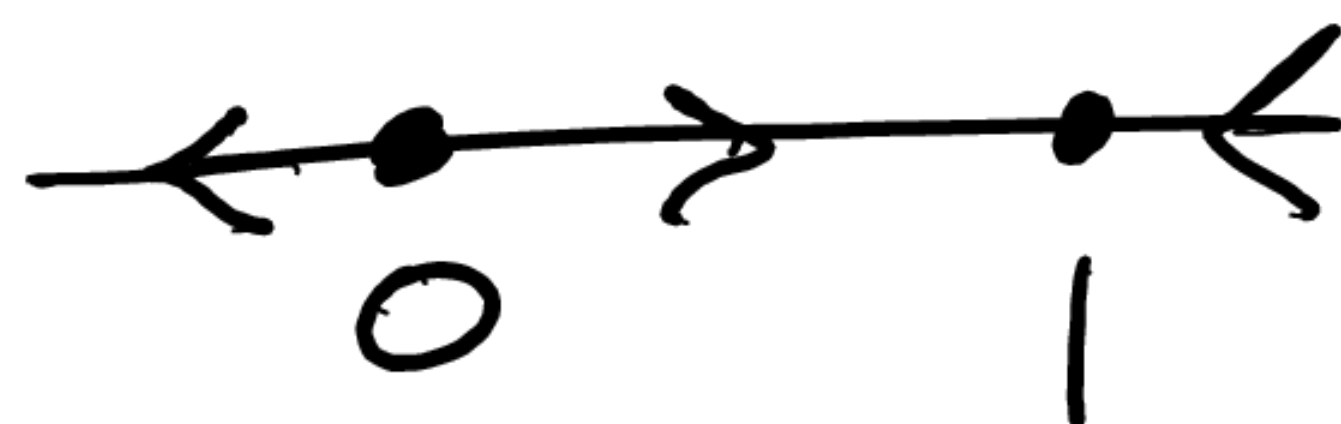
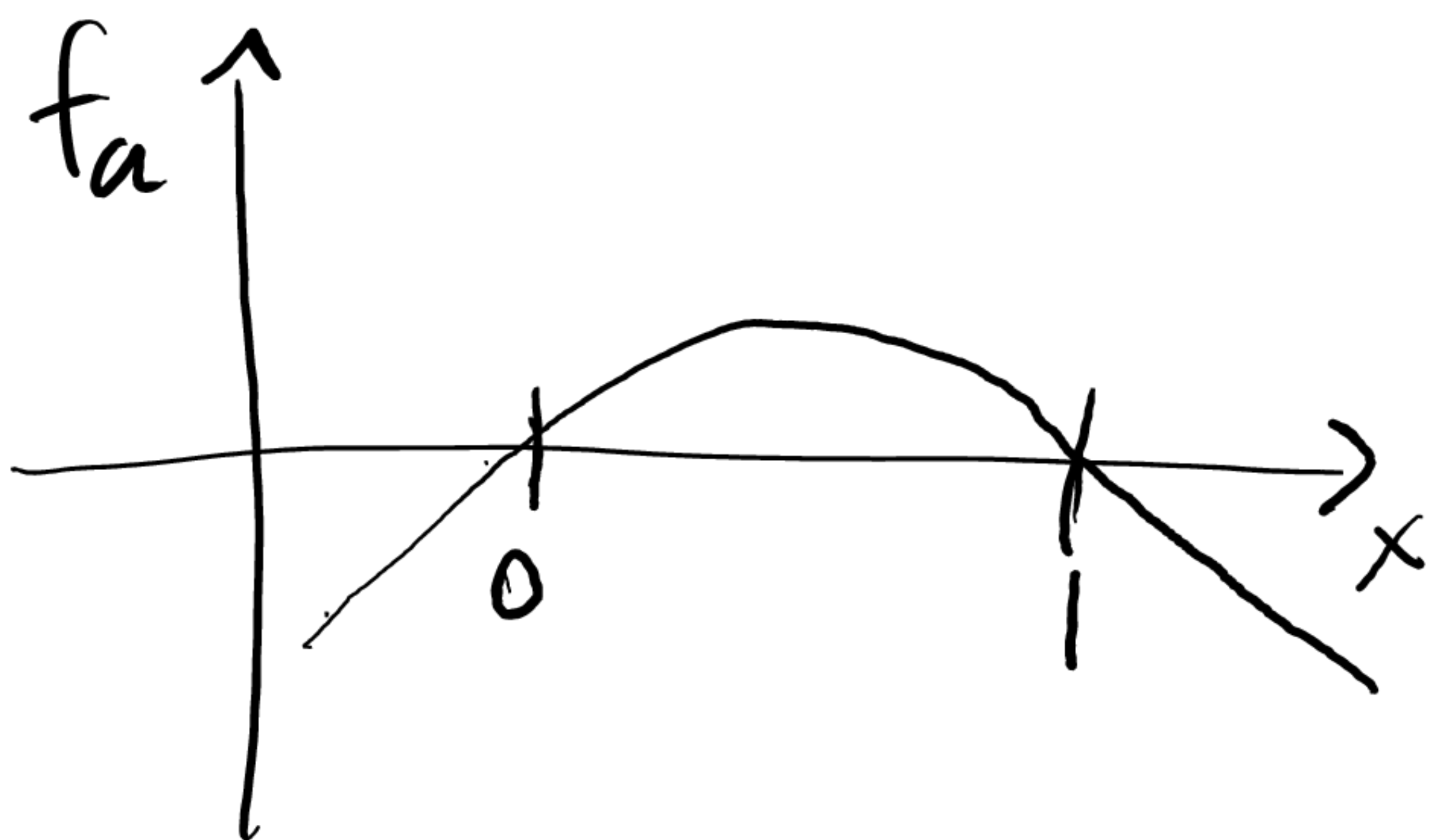
Normalization Consider $Y(t) = \frac{x}{N}$.

Then $Y'(t) = aY(1-Y) =: f_a(Y)$,

so WLOG we assume $N=1$.

EP: 0 and 1

Phase line



0 is unstable, 1 is A-stable.

SoV: Seek solution $x(t) \in (0,1)$:

$$\int \frac{1}{f_a(x)} dx = \int 1 dt$$

$$\Rightarrow \frac{1}{a} \int \frac{1}{x} + \frac{1}{1-x} dx = t + C$$

$$\ln\left(\frac{x}{1-x}\right) = at + C$$

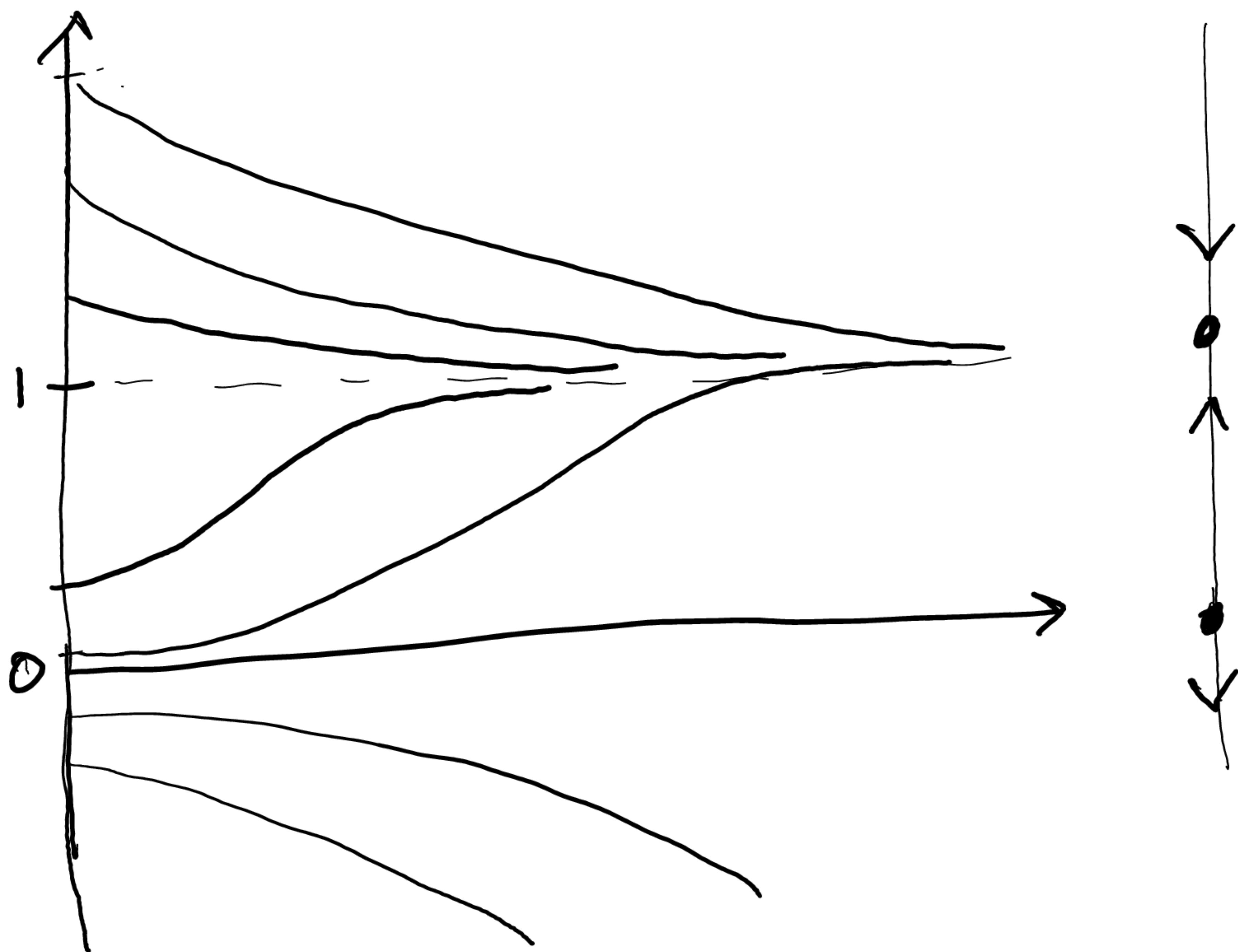
$$x(t) = \frac{ke^{at}}{1+ke^{at}} \quad \text{constant } k > 0.$$

Formula holds in general for $x(t) \geq 0$, with $k \in \mathbb{R}$.

Given $x(0) = x_0$, solve for k to get

$$x(t) = \frac{x_0 e^{at}}{1 + x_0(e^{at} - 1)}$$

Solution curves



Constant harvesting

$$x' = x(1-x) - h =: f_h(x)$$

with $h > 0$ models population with growth rate $a=1$, "ideal" size 1, and constant harvest rate.

EP: x^* where $f_h(x^*) = 0$.

Algebraic approach:

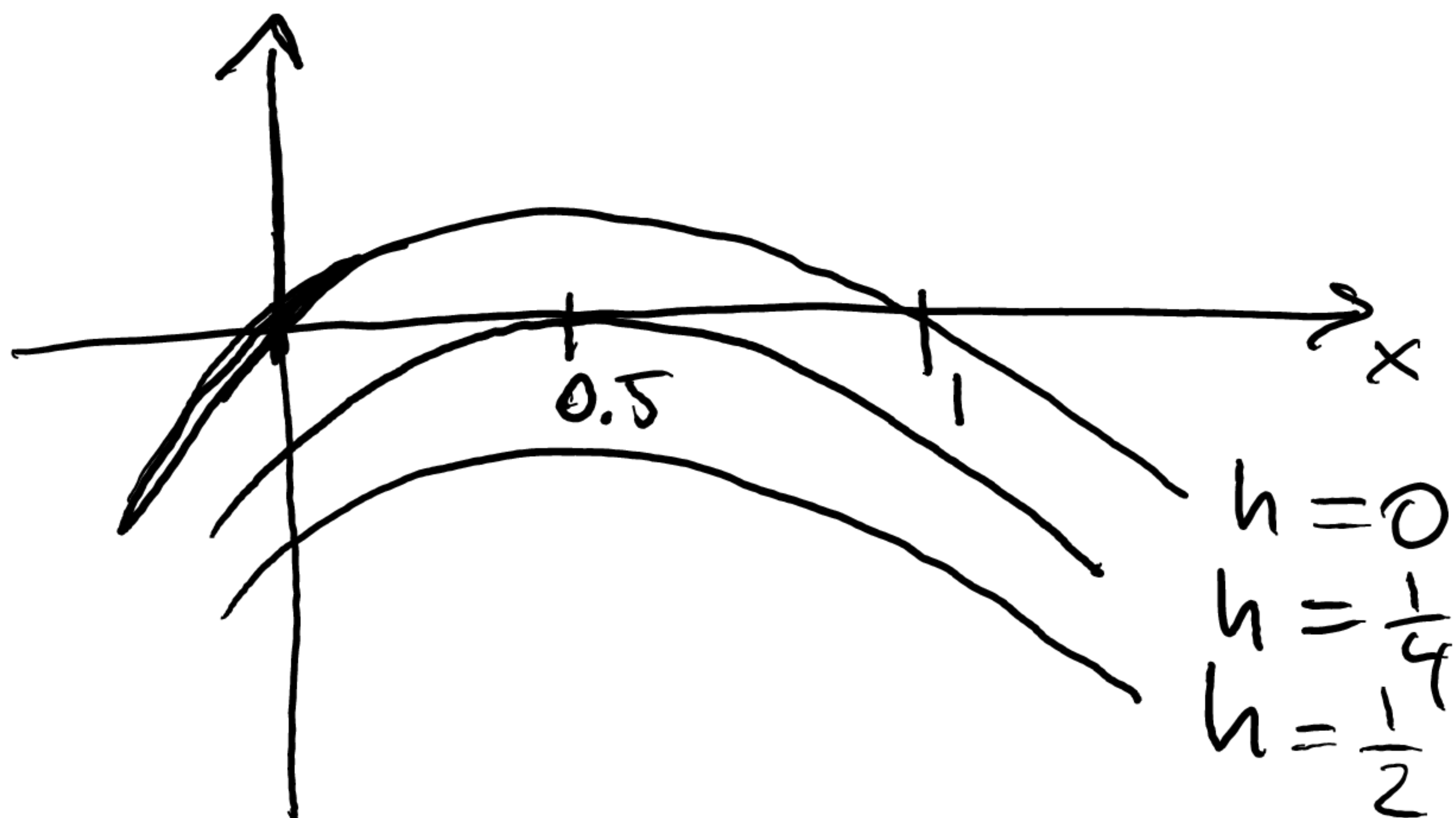
$x^2 - x + h = 0$, yields EP

$$x^* = \begin{cases} \frac{1 \pm \sqrt{1-4h}}{2} \\ \frac{1}{2} \end{cases},$$

$$h \in (0, \frac{1}{4})$$

$$h = \frac{1}{4}$$

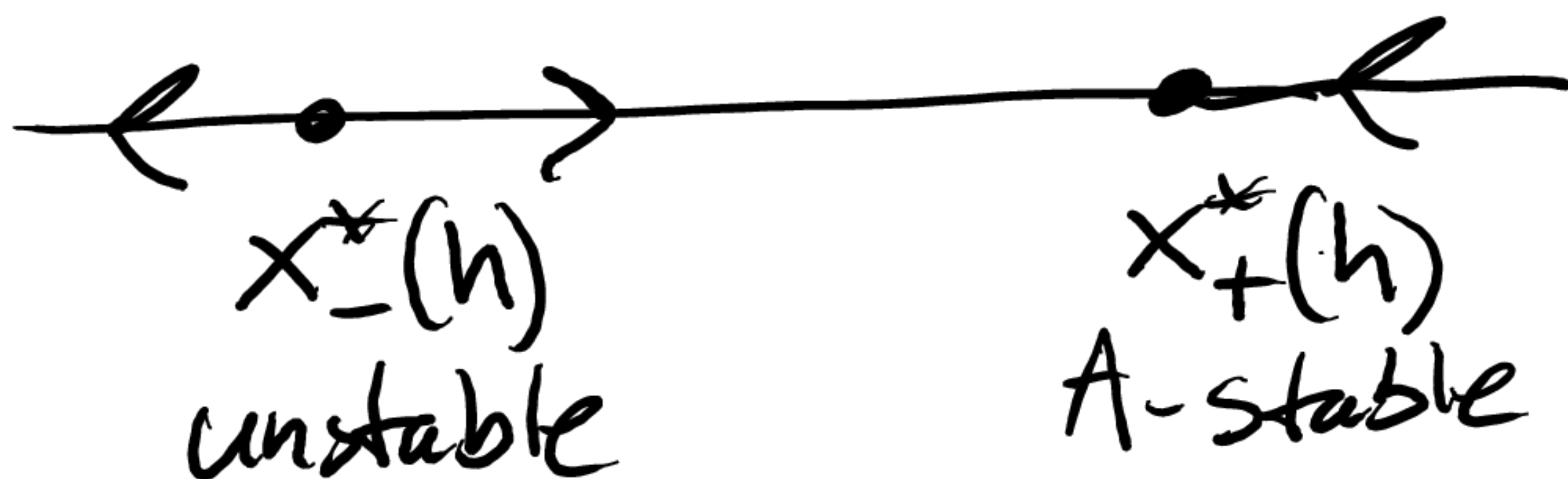
Geometric



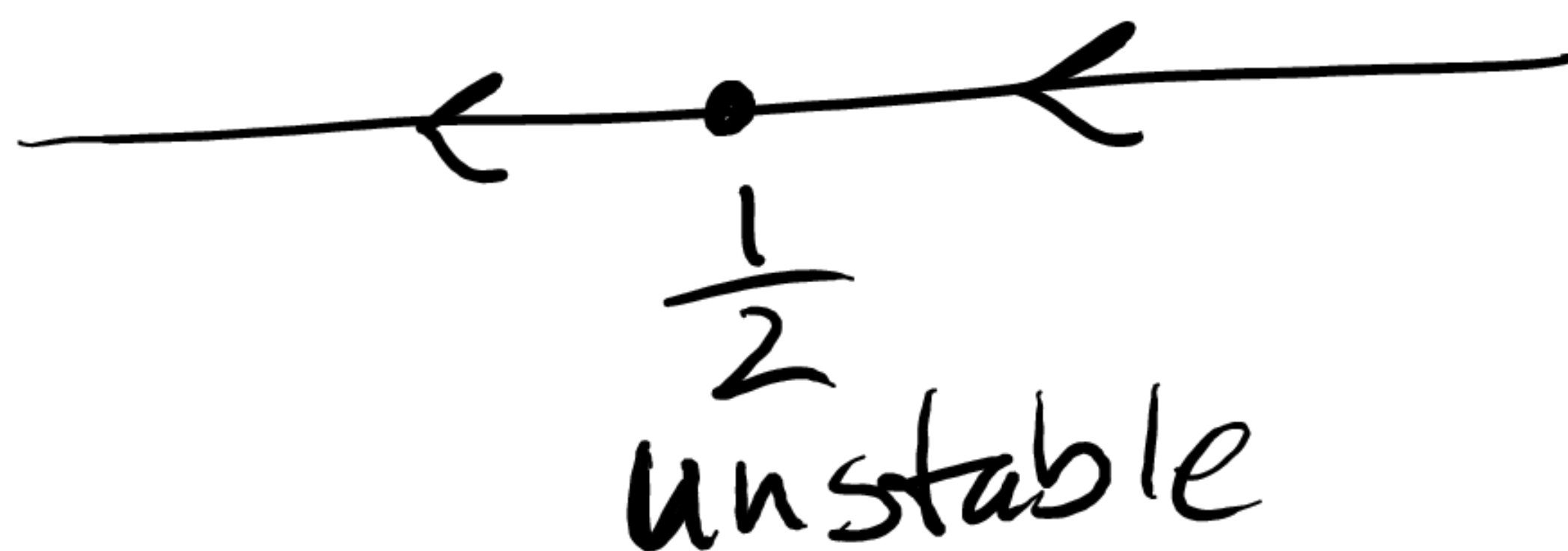
Phase lines

$$x_{\pm}^*(h) = \frac{1 \pm \sqrt{1-4h}}{2}$$

$$0 < h < \frac{1}{4}$$



$$h = \frac{1}{4}$$



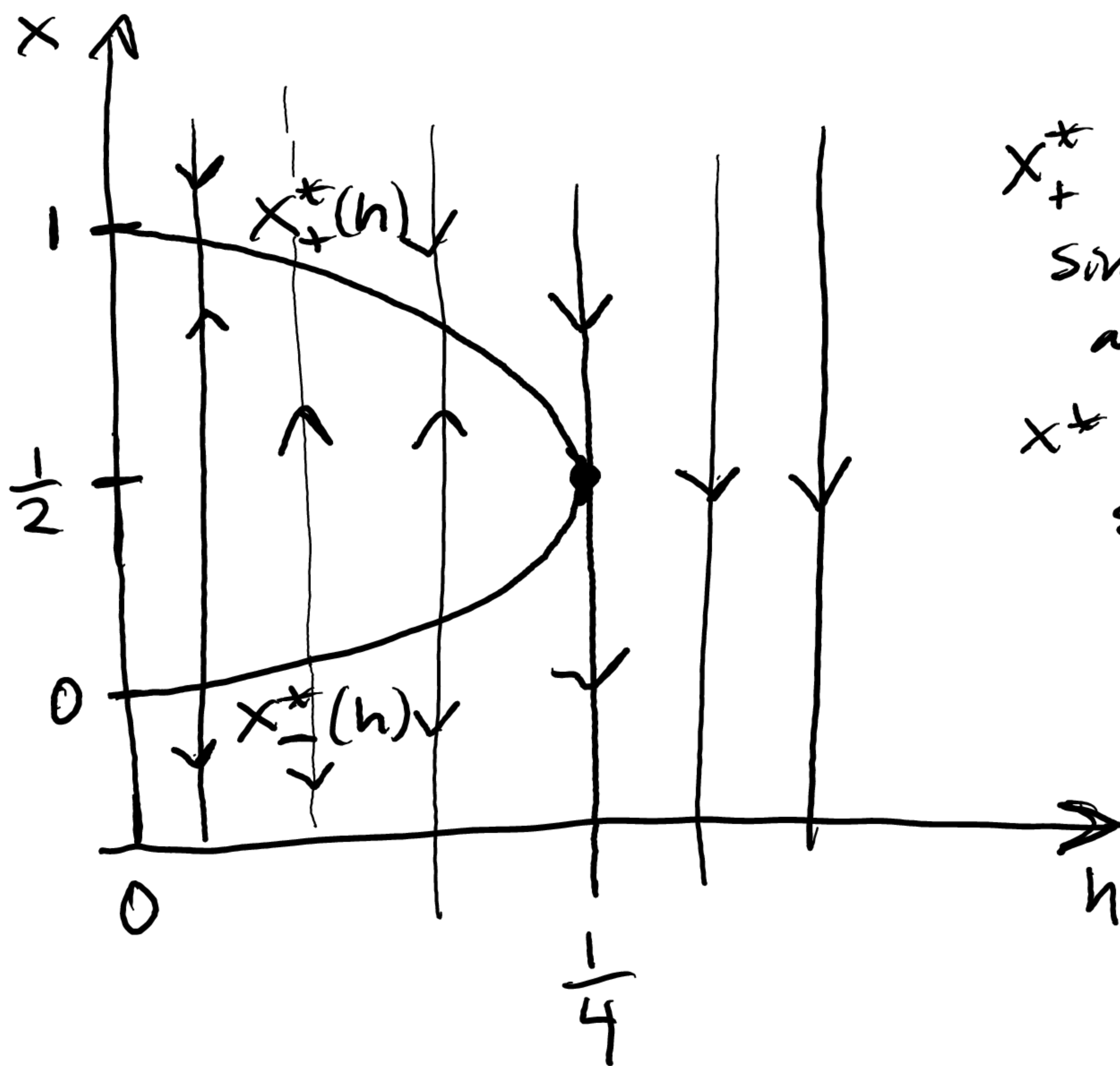
$$h > \frac{1}{4}$$



For $h > \frac{1}{4}$, we see (without solving it exactly) that $\lim_{t \rightarrow \infty} x(t) = -\infty$ (as then $f_h < 0$).

Sustainable harvesting: $h \leq \frac{1}{4}$.

Bifurcation diagram (how do EP depend on h ?)



x_+^* change from sink to neither at $h = \frac{1}{4}$.

x_-^* change from source to neither at $h = \frac{1}{4}$.

Exercise Let $h \in (0, \frac{1}{4})$ and $x_0 \in (x_-^*, x_+^*)$. Note that

$$x' = f_h(x) = -(x - x_+^*)(x - x_-^*).$$

Solve ODE for all times s, t . $x(t) \in (x_-^*, x_+^*)$
using SolV.

Exercise 1.1

- Find general solution to $x' = ax + 3 = f_a(x)$
 $a \in \mathbb{R}$
- Determine EP for different values of a .
- Classify if EP are sources or sinks.

Solution method $a = 0$, $x' = 3 \Rightarrow x = 3t + C$

$$a \neq 0: x' e^{-at} - a x e^{-at} = 3 e^{-at}$$

$$\Rightarrow \frac{d}{dt}(x e^{-at}) = 3 e^{-at} \quad \Big| \int dt$$

$$\Rightarrow x(t)e^{-at} = -\frac{3e^{-at}}{a} + C$$

$$x(t) = -\frac{3}{a} + Ce^{at}$$

$$EP: ax + 3 = 0, \quad x^* = -\frac{3}{a} \text{ if } a \neq 0$$

