

Lecture 3 MAT3440, 26/1

Recall constant harvest

$$x' = x(1-x) - h \quad \text{with } h > 0,$$

only sustainable when $h \in (0, \frac{1}{4})$

and $x_0 \geq x_-^*(h)$.

Generalization to non-autonomous ODE:

$$x' = x(1-x) - \underbrace{h(1 + \sin(2\pi t))}_{\text{const Periodic harvesting}} = f_h(x, t)$$

Nice feature: $f_h(x, t) = f_h(x, t+1) \quad \forall x, t$

Challenges: Non-autonomous ODE.

Even if x^* is s.t. $f(x^*, t) = 0$
some $t \in \mathbb{R}$, it may be that $f(x^*, s) \neq 0$,
so stability theory and EP do not
generalize directly.

General property: Suppose $x' = f(x, t)$
with $f(x, t) = f(x, t+1) \quad \forall x, t$.

Let x_1 solve

$$x_1' = f(x_1, t) \quad t \in [0, 1] \quad (1)$$

$$x_1(0) = x_0$$

and x_2 solve

$$x_2' = f(x_2, t) \quad t \in [0, 1] \quad (2)$$

$$x_2(0) = x_1(1)$$

$$\text{Then } x_3(t) := \begin{cases} x_1(t) & t \in [0, 1] \\ x_2(t-1) & t \in (1, 2] \end{cases}$$

solves

$$x_3' = f(x_3, t) \quad t \in [0, 2]$$

$$x_3(0) = x_0$$

as for $t \in (1, 2]$,

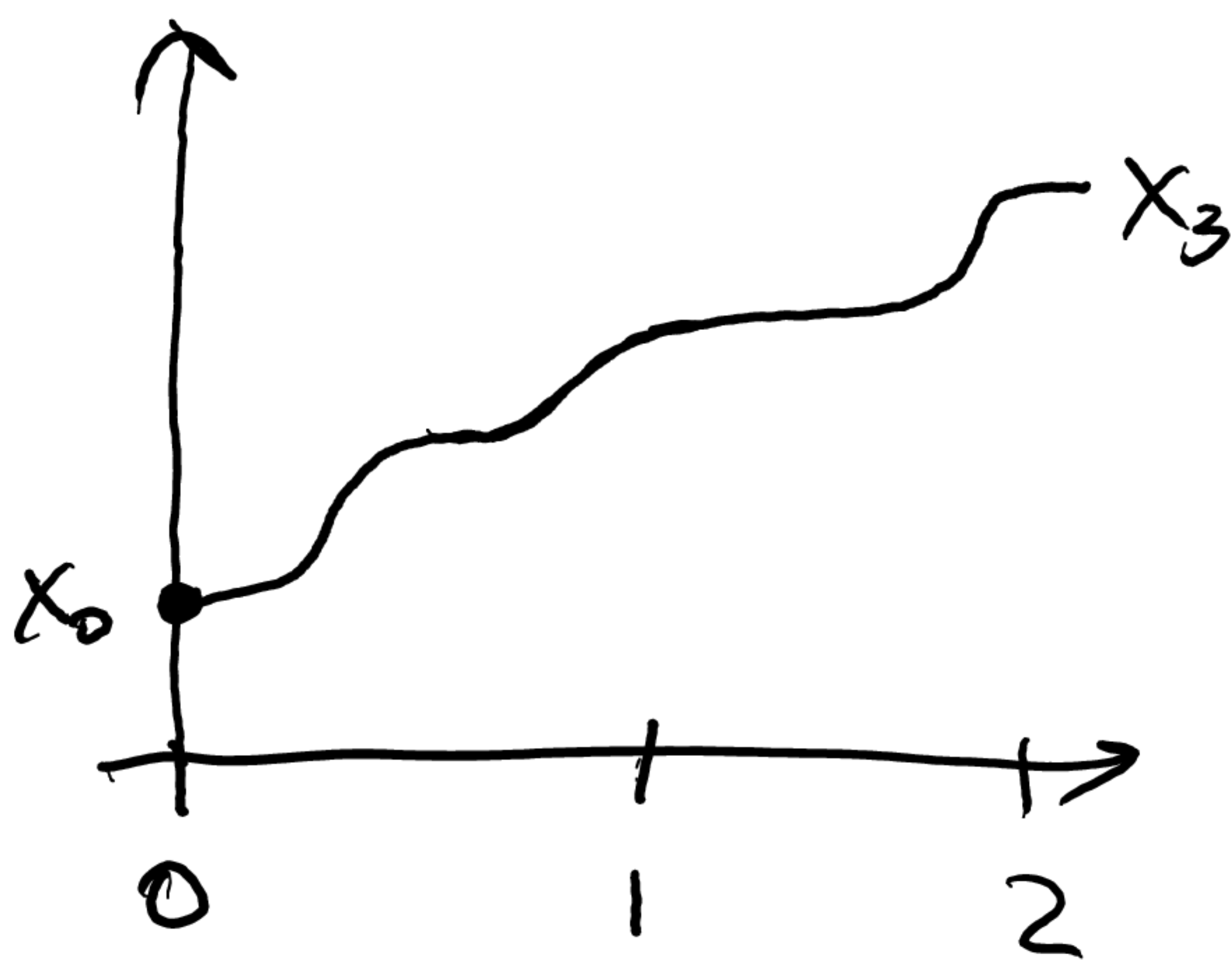
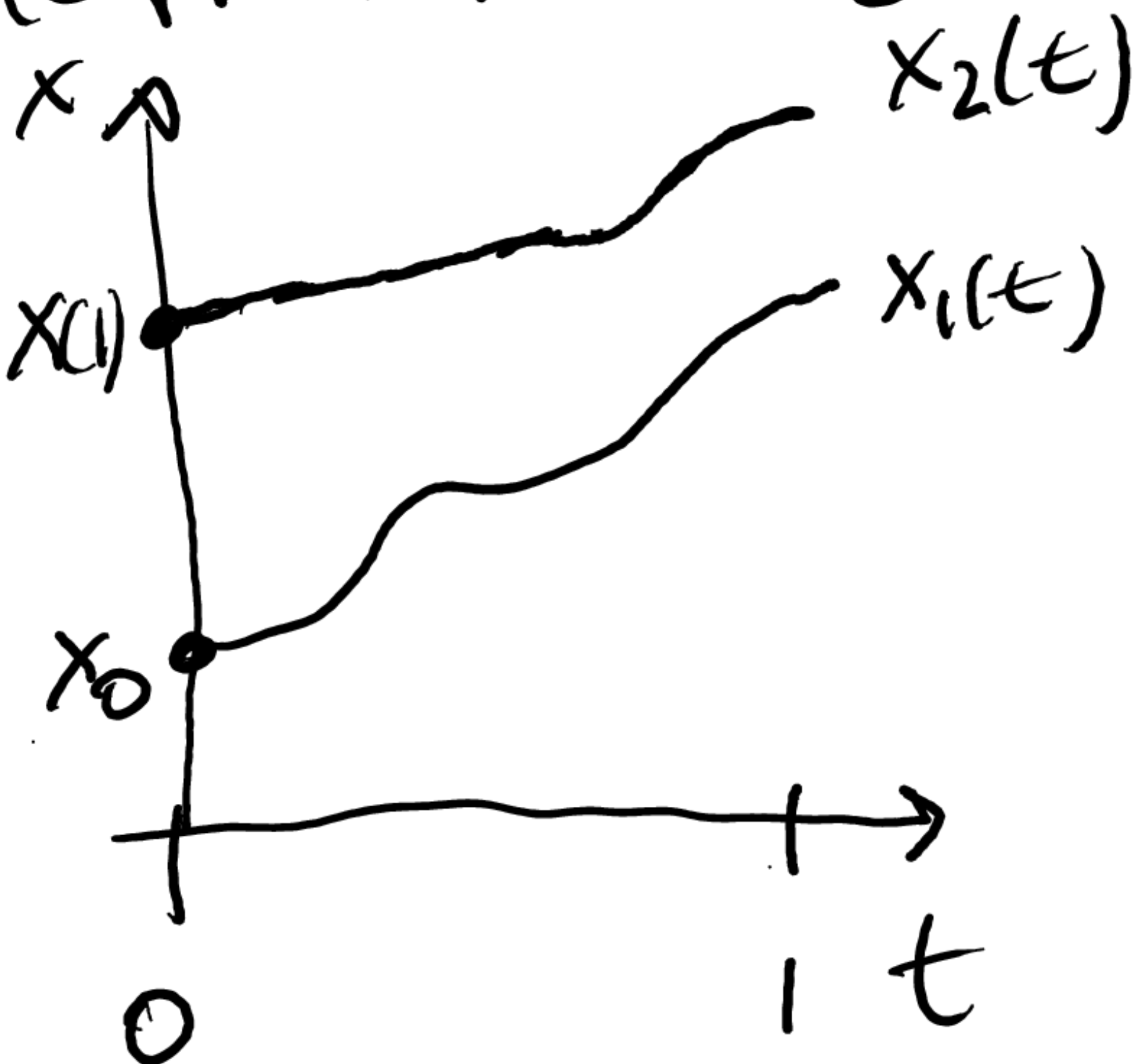
$$x_3'(t) = x_2'(t-1) = f(x_2(t-1), t-1)$$

$$= f(x_2(t-1), t-1) = f(x_3, t).$$

Conclusion: If we can solve ODE (1) for any x_0 (over $t \in [0, 1]$), then by iterated plugging in (3), we obtain a solution to
$$x' = f(x, t) \quad \boxed{t \geq 0}$$

$$x(0) = x_0$$

(Often the only solution, we assume that here.)



And If $x_1(1) = x_0$, then the solution will be periodic:



Def Let $\phi(t, x_0) := X(t)$, where $X(t)$ solves $X' = f(X, t)$
 $X(0) = x_0$.

ϕ is called the flow of the ODE.

Observe $\frac{\partial}{\partial t} \phi(t, x_0) = X'(t) = f(X(t), t)$
 $= f(\phi(t, x_0), t)$

$$\Rightarrow \phi(t, x_0) = \underbrace{\phi(0, x_0)}_{=x_0} + \int_0^t f(\phi(s, x_0), s) ds \quad (3)$$

Property of flow:

$$\phi(t+s, x_0) \stackrel{(3)}{=} \phi(t, \phi(s, x_0)) \quad (4)$$

For above 1-periodic problem,

$$X(1) = x_0 \Rightarrow \phi(1, x_0) = x_0 \text{ and}$$

$$X(2) = \phi(2, x_0) \stackrel{(4)}{=} \phi(1, \phi(1, x_0)) = \phi(1, x_0) = x_0$$

By iteration, $X(n) = x_0 \quad \forall n \in \mathbb{N}$.

For any $t \geq 0$, we can write $t = n + s$ for some integer $n \geq 0$ and $s \in [0, 1)$, and

$$\begin{aligned} X(t) &= \phi(n+s, x_0) \stackrel{(4)}{=} \phi(s, \phi(n, x_0)) \\ &= \phi(s, x_0) = X(s) \end{aligned}$$

Similarly $X(t+1) = X(s)$.

Hence, $X(t+1) = X(t) \quad \forall t \geq 0$.

The Poincaré map

$P(x_0) := \phi(1, x_0)$ is important for ODEs where $f(x, t+1) = f(x, t)$.

Every x_0 s.t. $P(x_0) = x_0$, produces a solution $X(t) = \phi(t, x_0)$ that is 1-periodic.

For stability analysis

$$\left. \begin{array}{l} x' = f(x, t) \\ x(0) = x_0 \end{array} \right\} \begin{array}{l} \text{1-Periodic} \\ (5) \end{array}$$

$$\left. \begin{array}{l} x' = f(x) \\ x(0) = x_0 \end{array} \right\} (6)$$

Fixed point (FP)

EP

$$\begin{array}{l} P(x^*) = x^* \\ \text{of discrete dynamics} \\ x_{n+1} = P(x_n) \end{array}$$

$$f(x^*) = 0$$

play similar roles.

Note: x^* EP of (6) and
 $f(x)$ trivially being 1-periodic in t
 $\Rightarrow x^*$ also FP $P(x^*) = x^*$.

Periodic harvesting

$$x' = x(1-x) - h(1 + \sin(2\pi t)) = f_h(x, t)$$

with $h > 0$, and f_h 1-periodic in t .

Q: Can we find $x_0 > 0$ s.t. $x(t)$ is periodic, and for what $h > 0$ is harvest sustainable?

Flow map: $\phi(t, x_0) = x_0 + \int_0^t f(\phi(x_0, s), s) ds$ (7)
with shorthand $f := f_h$.

Poincaré map: $P(x_0) = \phi(1, x_0)$

Solution $x(t) = \phi(t, x_0)$ is periodic if $P(x_0) = x_0$.

Taking partial derivative $\frac{\partial}{\partial x_0}$ in (7),

$$\phi_{x_0}(t, x_0) = 1 + \int_0^t \frac{\partial f}{\partial x}(\phi(x_0, s), s) \cdot \phi_{x_0}(x_0, s) ds$$

where $f_x(x, t) := \lim_{h \rightarrow 0} \frac{f(x+h, t) - f(x, t)}{h}$.

Let $z(t) := \phi_{x_0}(t, x_0)$ (i.e., $z(t, x_0)$). Then

$$z(t) = 1 + \int_0^t f_x(\phi(s, x_0), s) \cdot z(s) \, ds$$

so z solves the ODE

$$z'(t) = f_x(\phi(t, x_0), t) z(t) \quad t \geq 0$$

$$z(0) = 1.$$

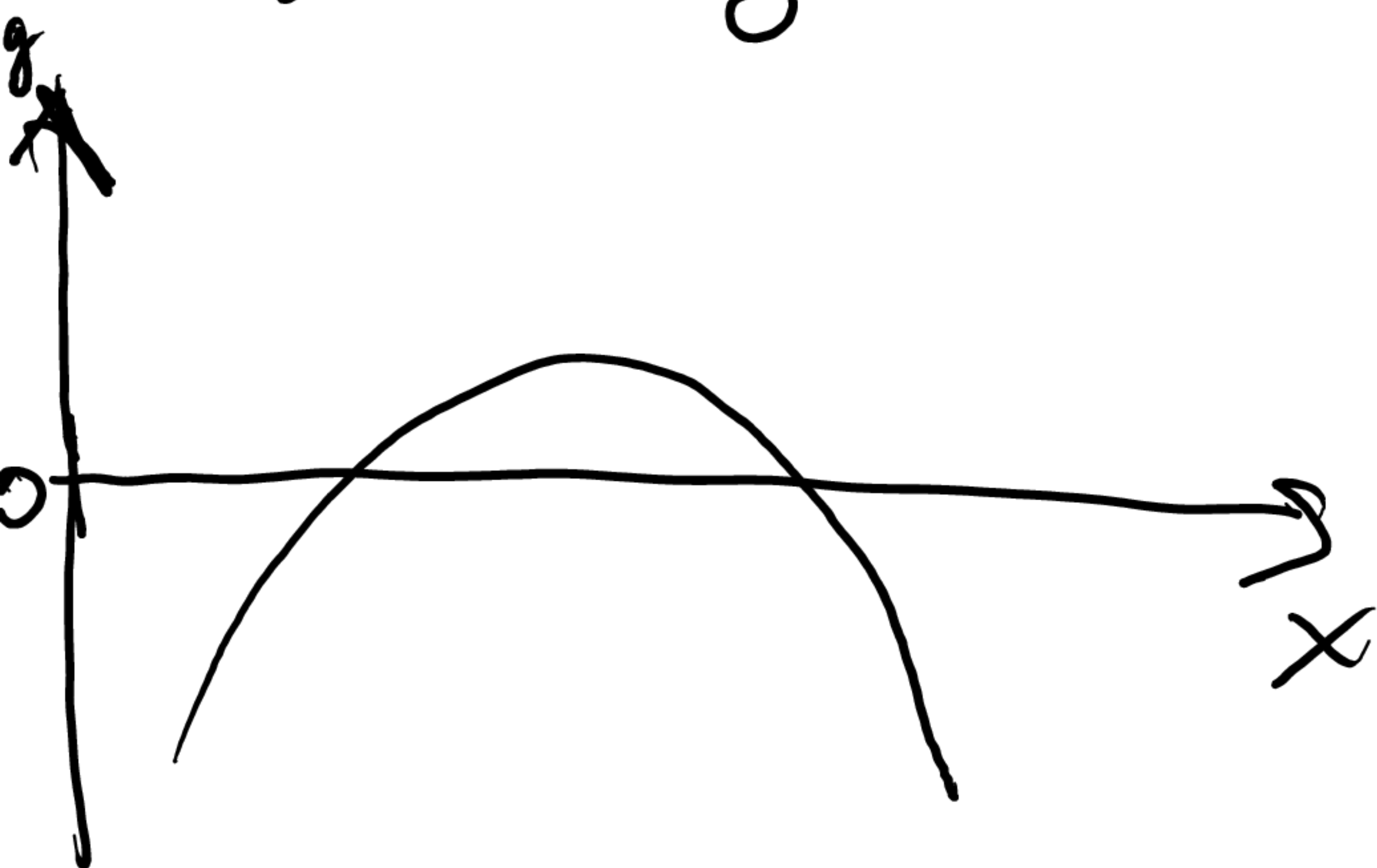
Integrating factor $e^{-\int_0^t f_x(\phi(s, x_0), s) \, ds}$
yields solution

$$z(t) = \exp\left(\int_0^t f_x(\phi(s, x_0), s) \, ds\right) \cdot \underbrace{1}_{z(0)}.$$

Note $p'(x_0) = \frac{\partial}{\partial x_0} \phi(1, x_0) = z(1) > 0 \quad \forall x_0$

Further $p''(x_0) = \frac{\partial}{\partial x_0} z(1, x_0) < 0 \quad \forall x_0$

Hence $P(x)$ is strictly convex, and
 so so is $g(x) := P(x) - x$.



$\Rightarrow g$ has at
 most 2 zeros

$\Rightarrow P$ has at
 most 2 FP.

$\Rightarrow$ ODE ()
 has at most 2
 1-periodic solutions.

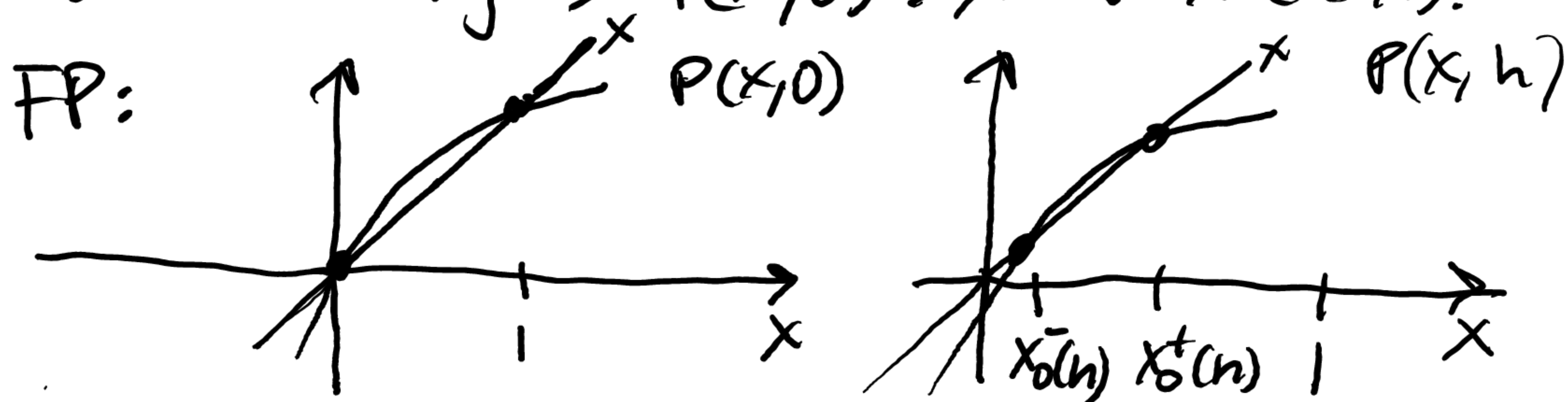
And if $x_0 < 0$

$$P(x_0, h) = x_0 + \int_0^1 \underbrace{x(s)(1-x(s))}_{< 0} - \underbrace{h(1+\sin(2\pi t))}_{< 0} ds$$

$$< x_0,$$

Similarly, $P(x_0, h) < x_0$ for $x_0 > 1$ So P
 can only have FP in $[0, 1]$.

For $h=0$, $P(x,0)$ has FP $x_0^- = 0$, $x_0^+ = 1$,
and concavity $\Rightarrow P(x,0) > x \quad \forall x \in (0,1)$.



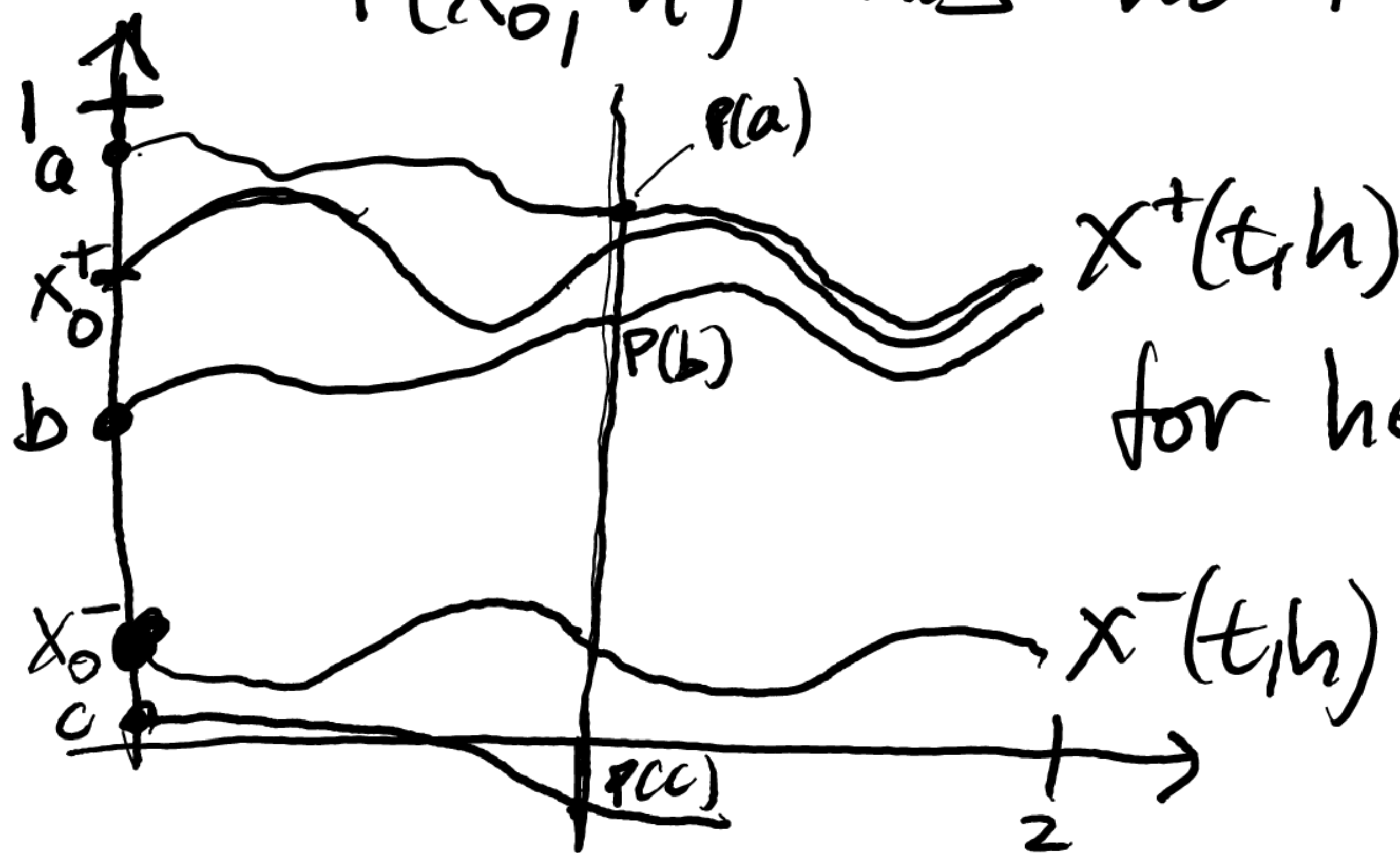
And since

$$\frac{\partial P(x, h)}{\partial h} = \int_0^1 -(1 + \sin(2\pi t)) dt = -1 < 0,$$

there exists a value $h^* > 0$ s.t.

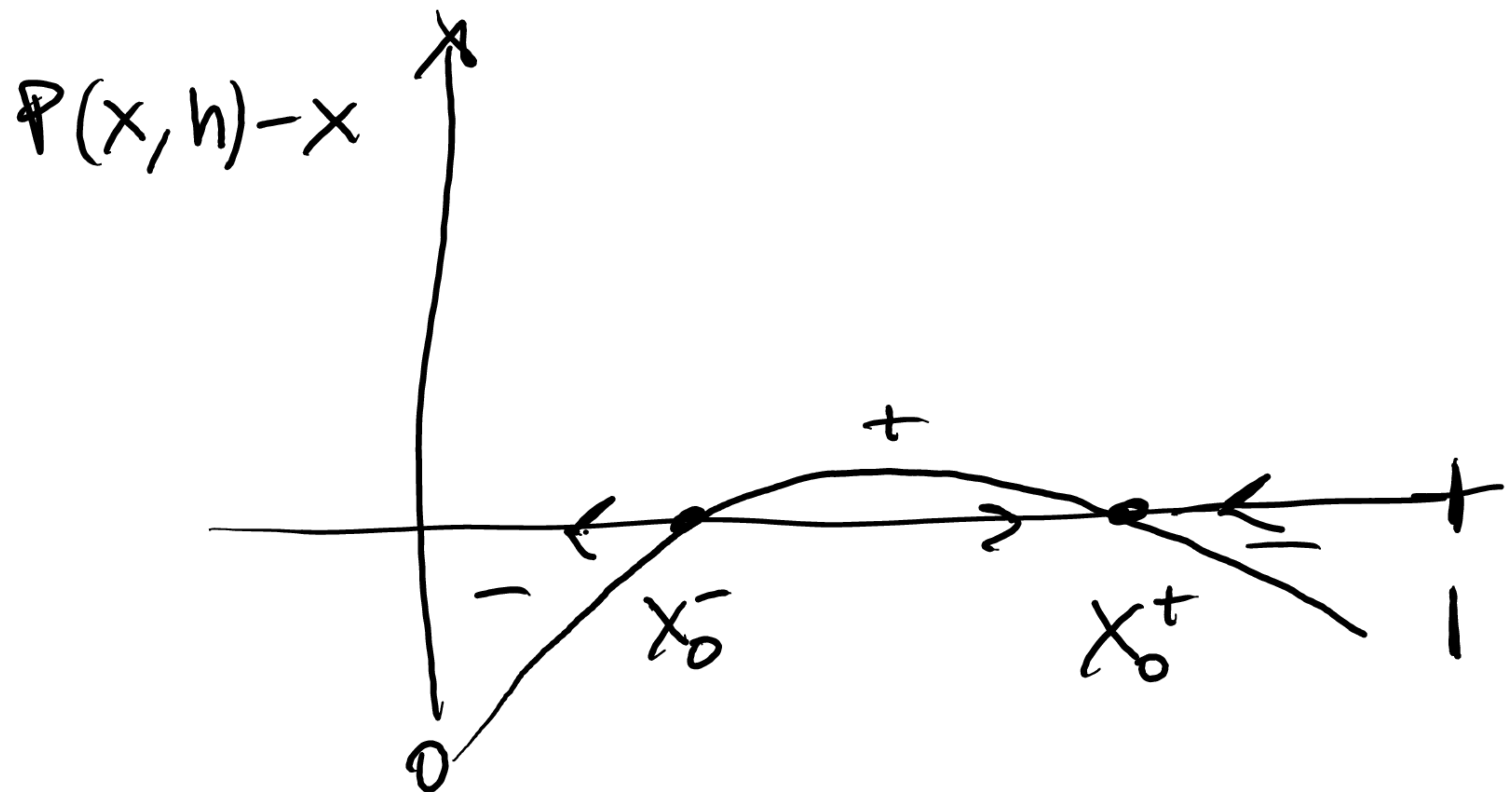
$P(x_0, h^*) = x$ for only one $x \in (0,1)$

and $P(x_0, h)$ has no FP for $h > h^*$.



Note
 $P(a, h) < a$
 $P(b, h) > b$
 $P(c, h) < c$
 for $h \in (0, h^*)$

Since $P(x, h)$ is concave and



FP x_0^- is unstable

while x_0^+ is stable.

Conclusion

Harvesting is stable if $h \in (0, h^*)$
and $x_0 \geq x_0^-(h)$.

Exercise 1.2 Find EP, classify as sinks, sources or neither, sketch phase line.

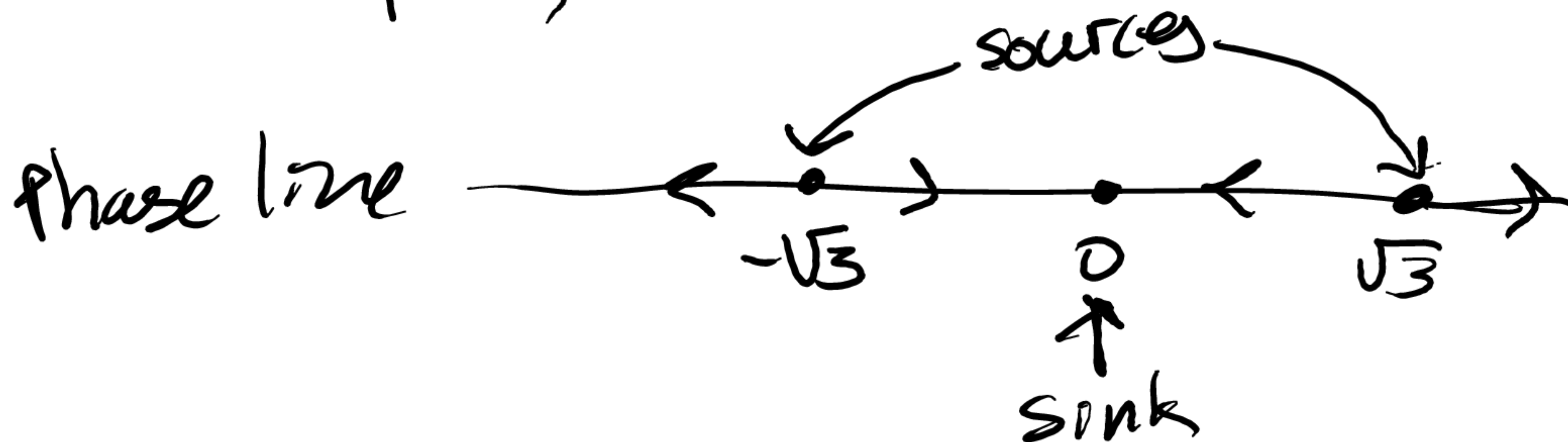
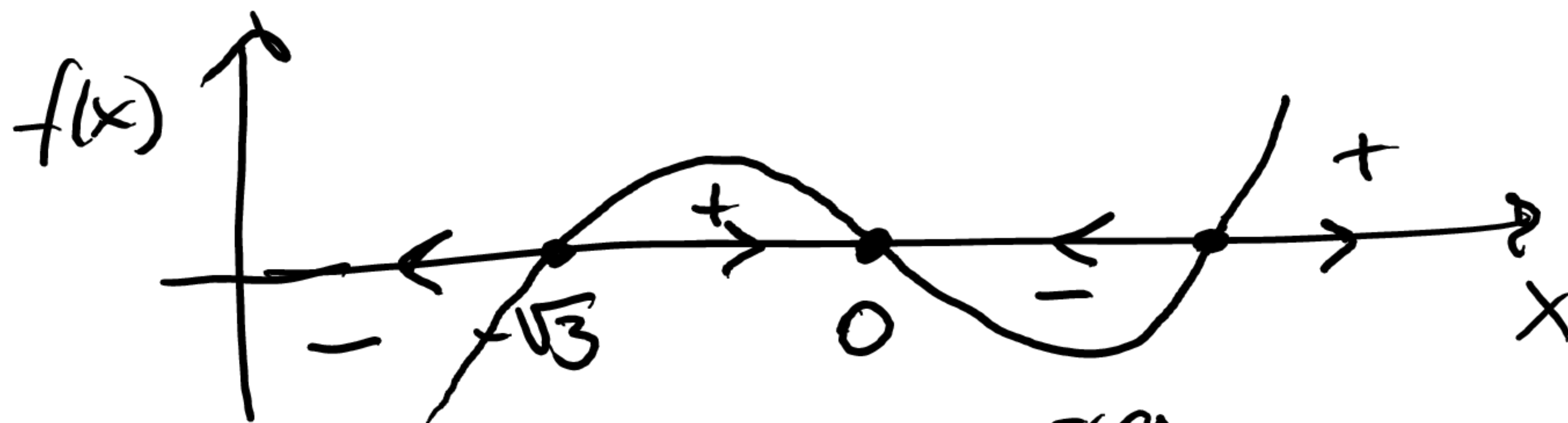
a) $x' = x^3 - 3x$

c) $x' = \cos x$

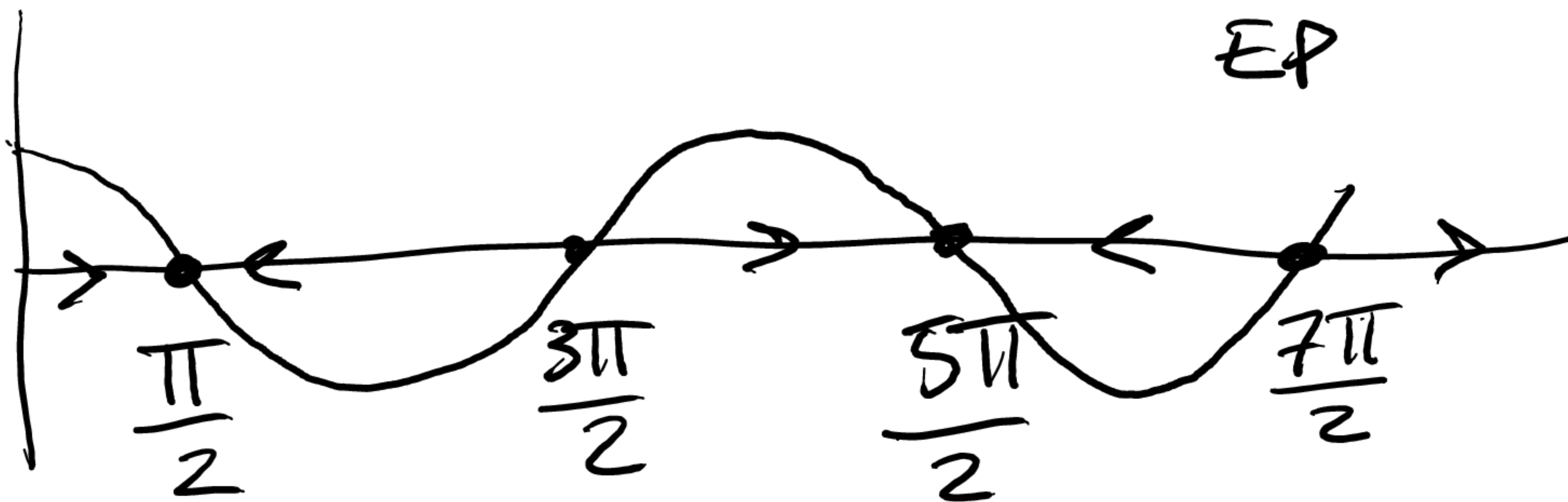
d) $x' = \sin^2 x$

a) $x^3 - 3x = x(x^2 - 3)$ has

EP/zeros: $0, \pm\sqrt{3}$



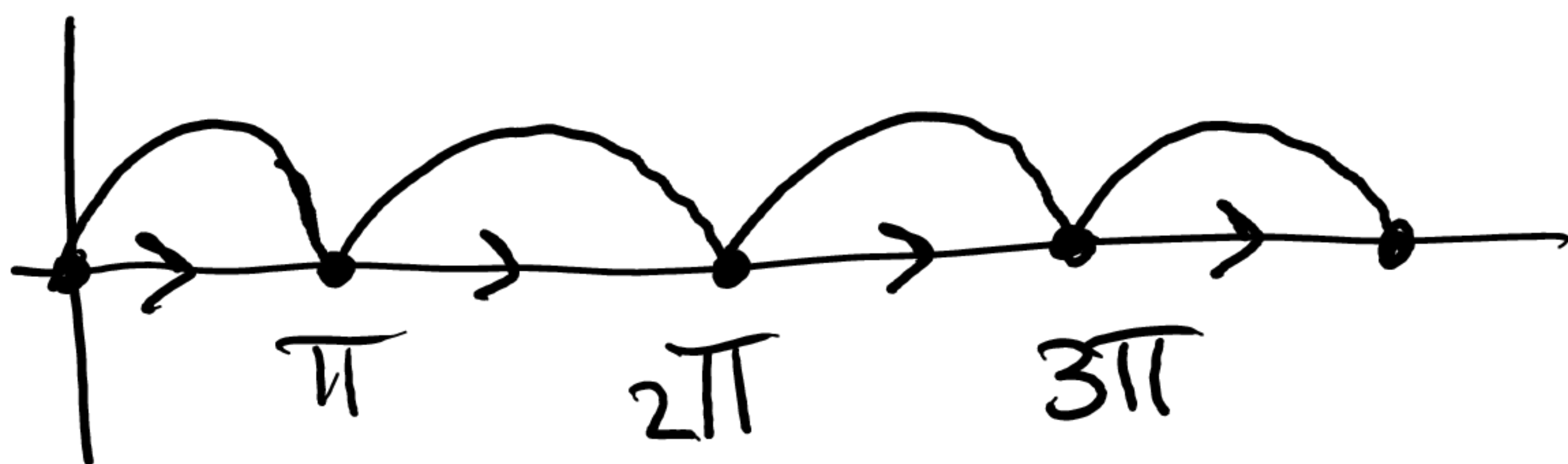
c) $\cos x = 0$ for $x = \underbrace{(2k+1)\frac{\pi}{2}}_{EP} \quad k \in \mathbb{Z}$



sinks: $(4k+1)\frac{\pi}{2} \quad k \in \mathbb{Z}$

sources: $(4k-1)\frac{\pi}{2} \quad k \in \mathbb{Z}$

d) $\sin^2 x = 0$ for $x = \underbrace{k\pi}_{EP} \quad k \in \mathbb{Z}$



0

All EP are neither sinks nor sources
(they are all unstable).