

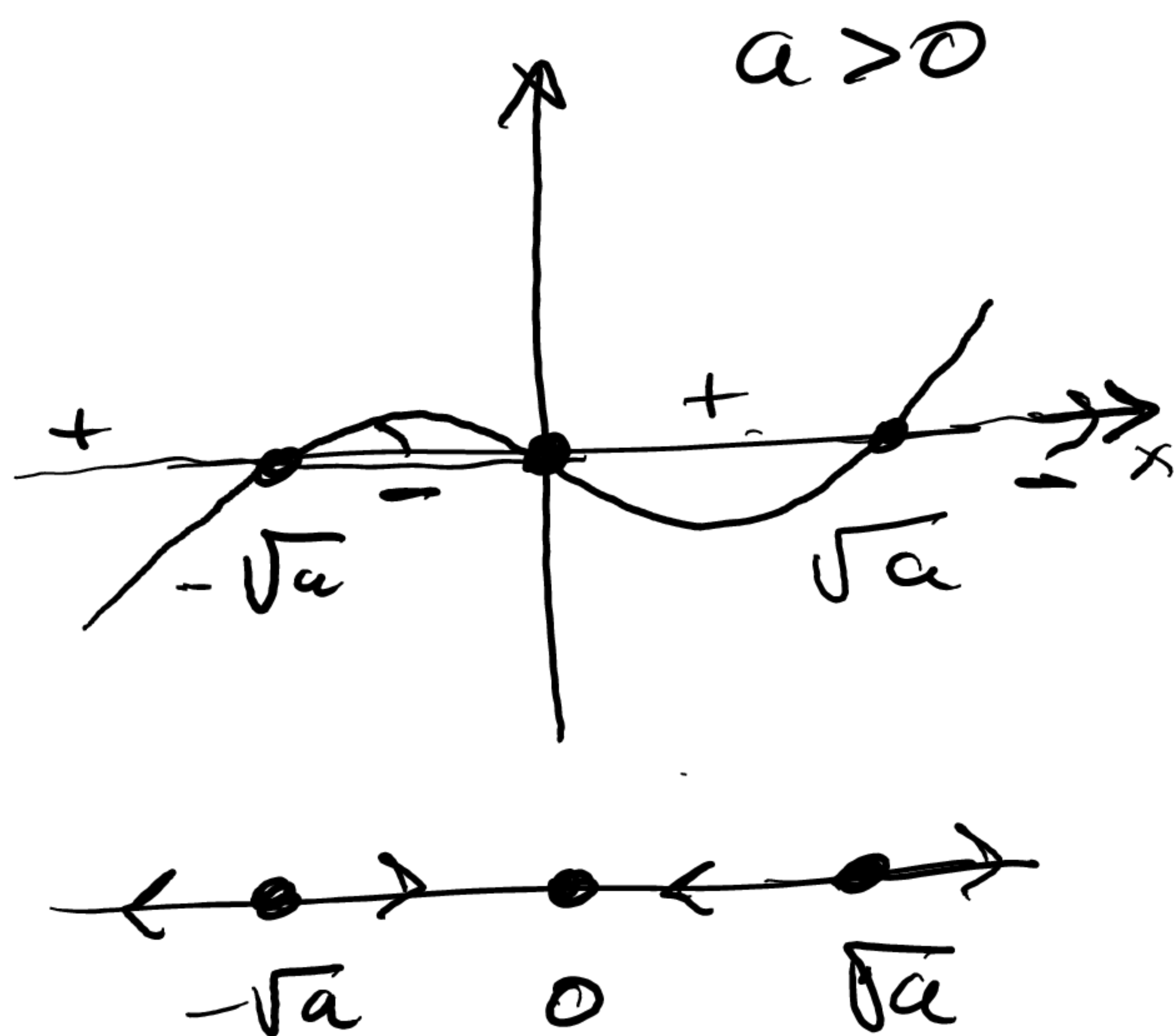
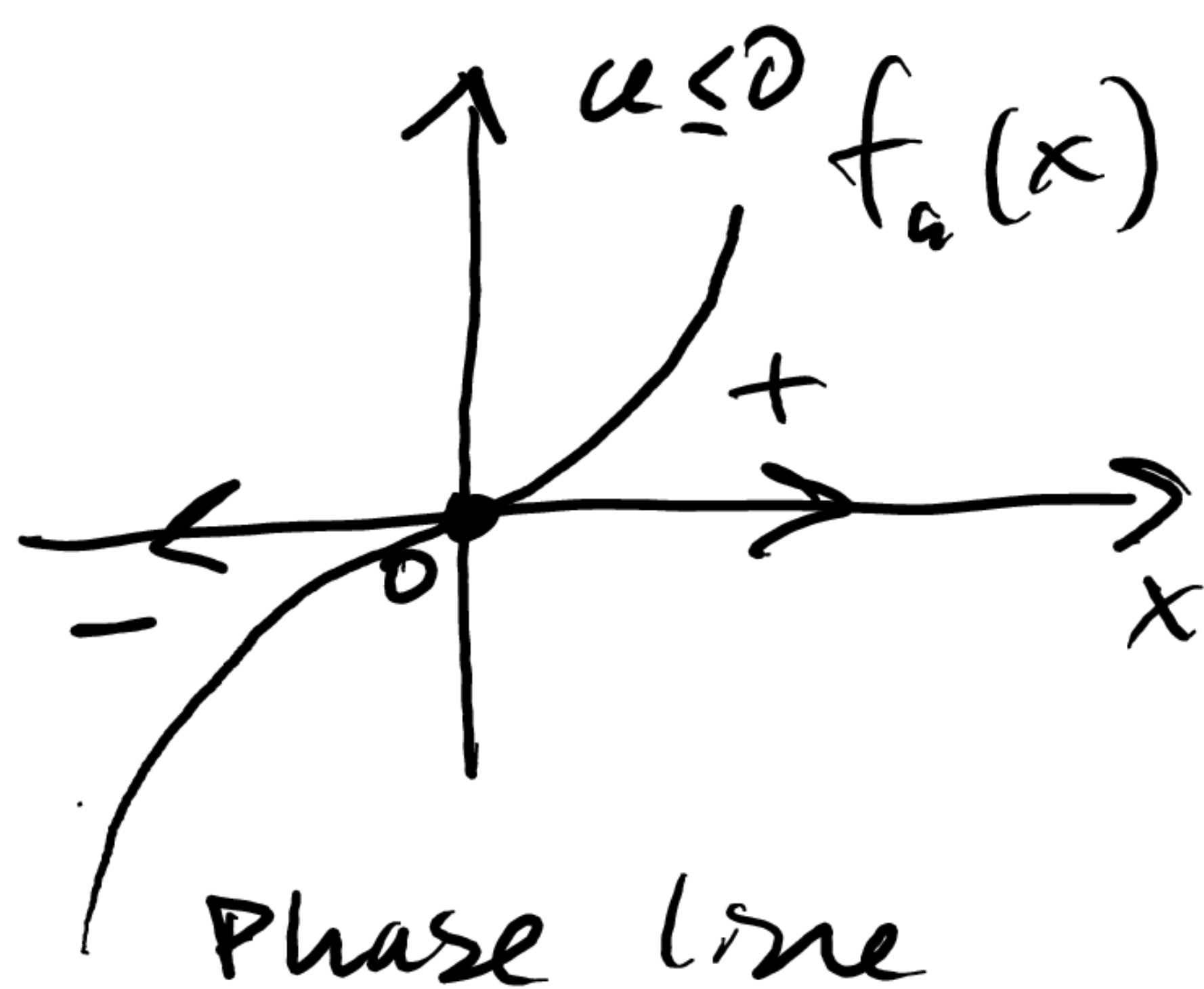
Exercises Chapter 1.

3b) sketch bifurcation diagram for

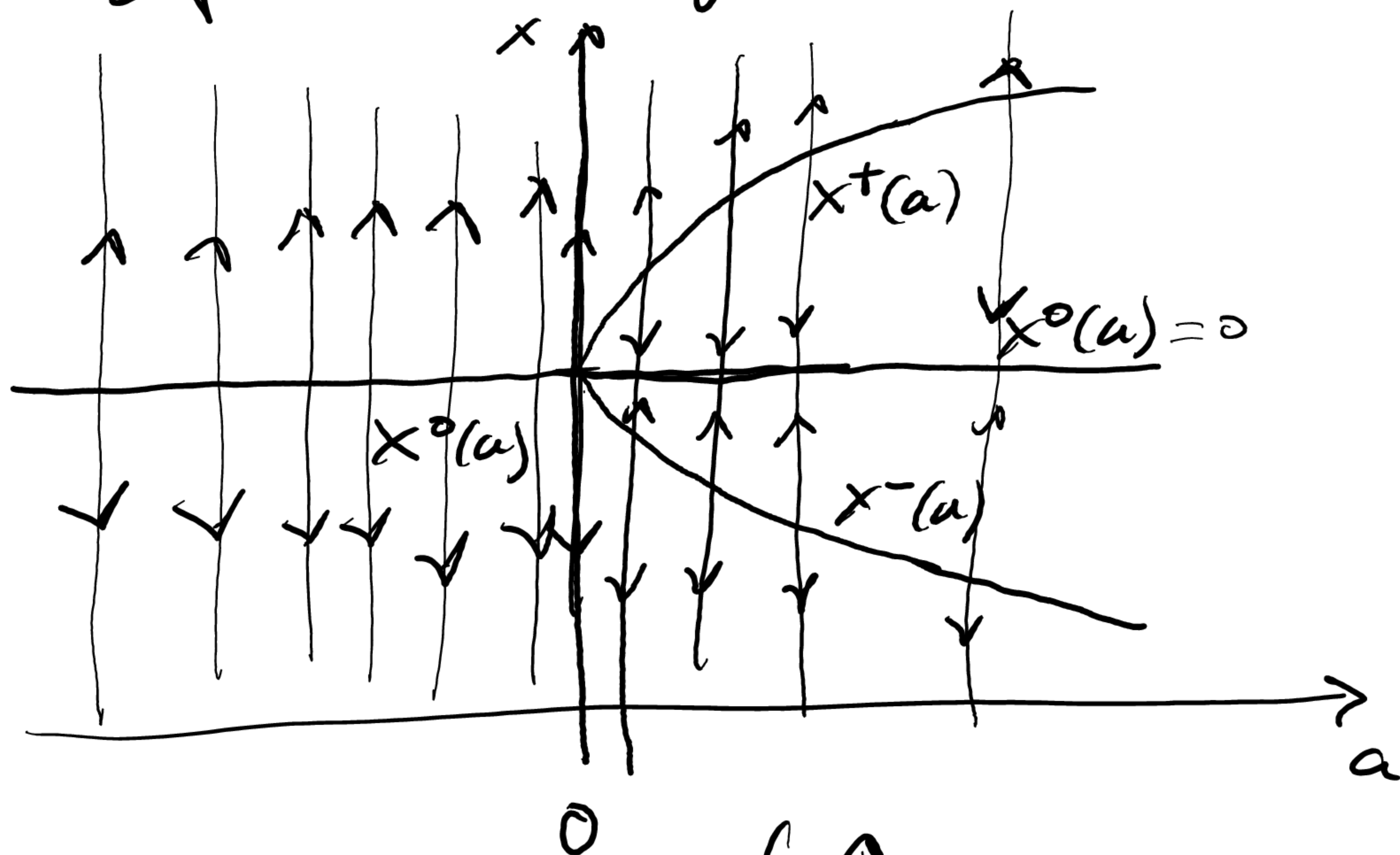
$$\underline{x' = x^3 - ax = f_a(x). \quad a \in \mathbb{R}}$$

$$\text{Sol: } x^3 - ax = x(x^2 - a)$$

has real-valued zeros

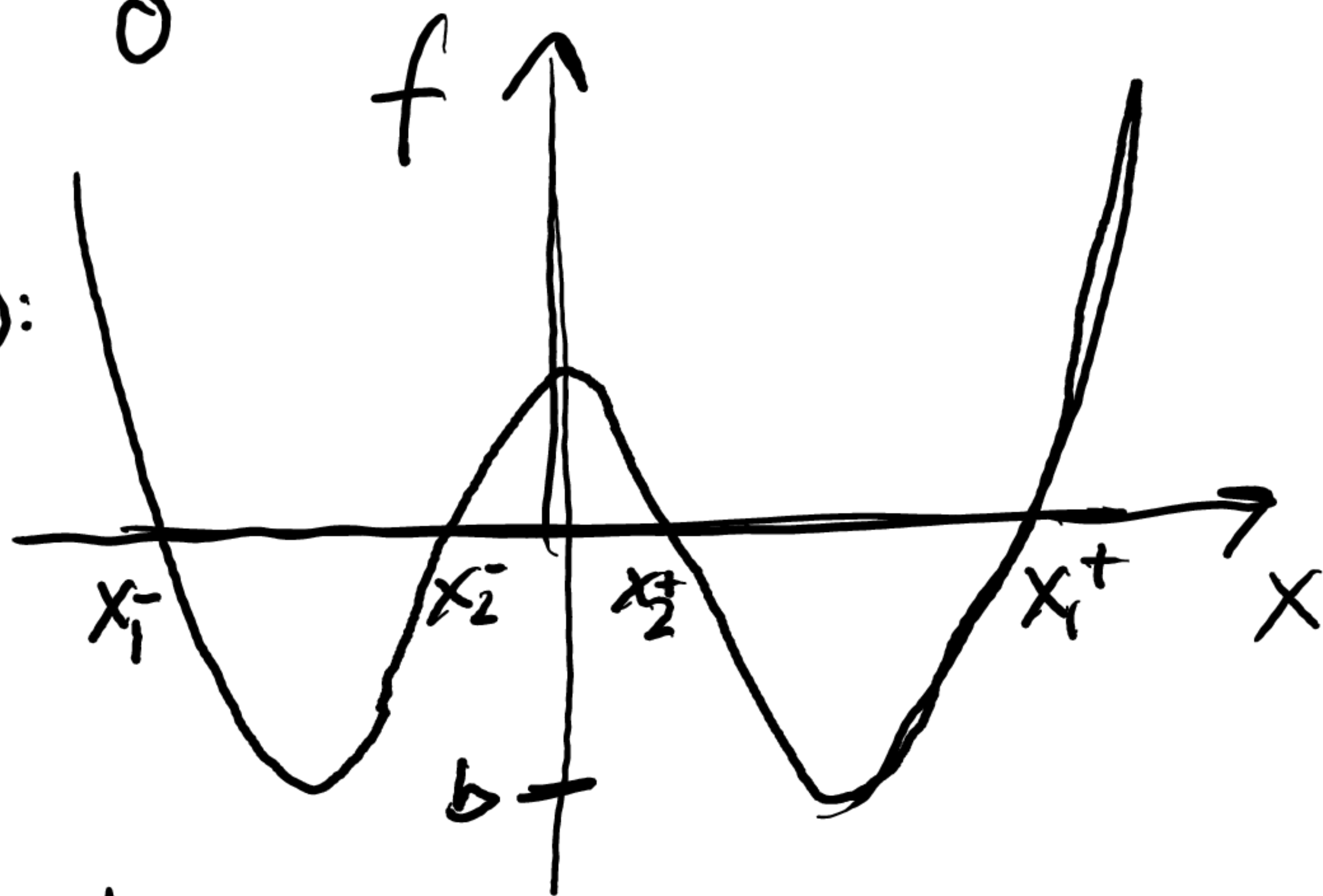
 $x_0 = 0$ if $a \leq 0$ and additionally $x^{\pm 1} = \pm \sqrt{a}$ if $a > 0$. These are the EP.

Bifurcation diagram



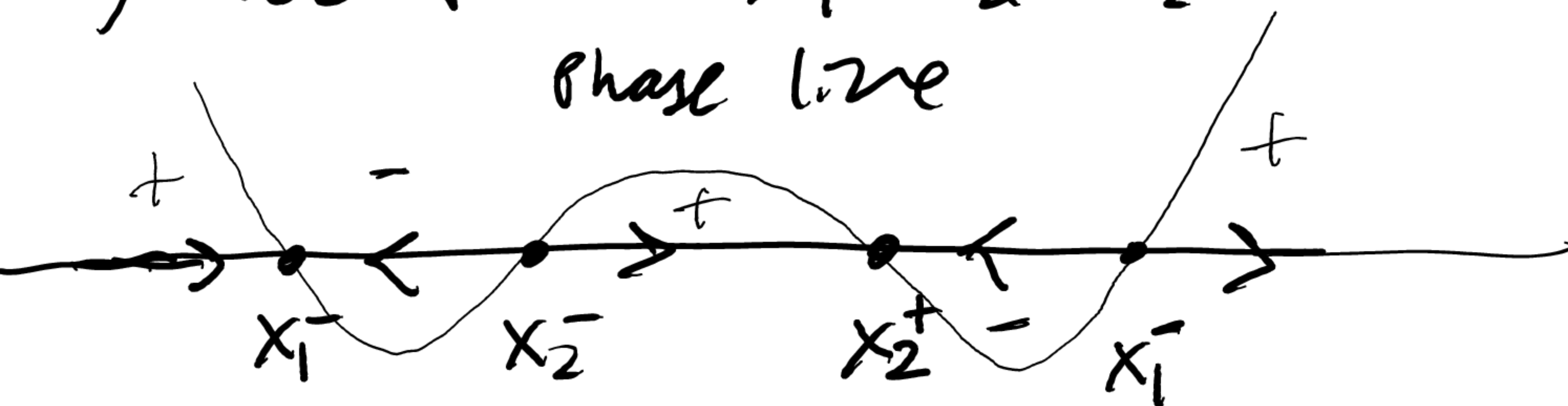
Given $f(x)$:

4.

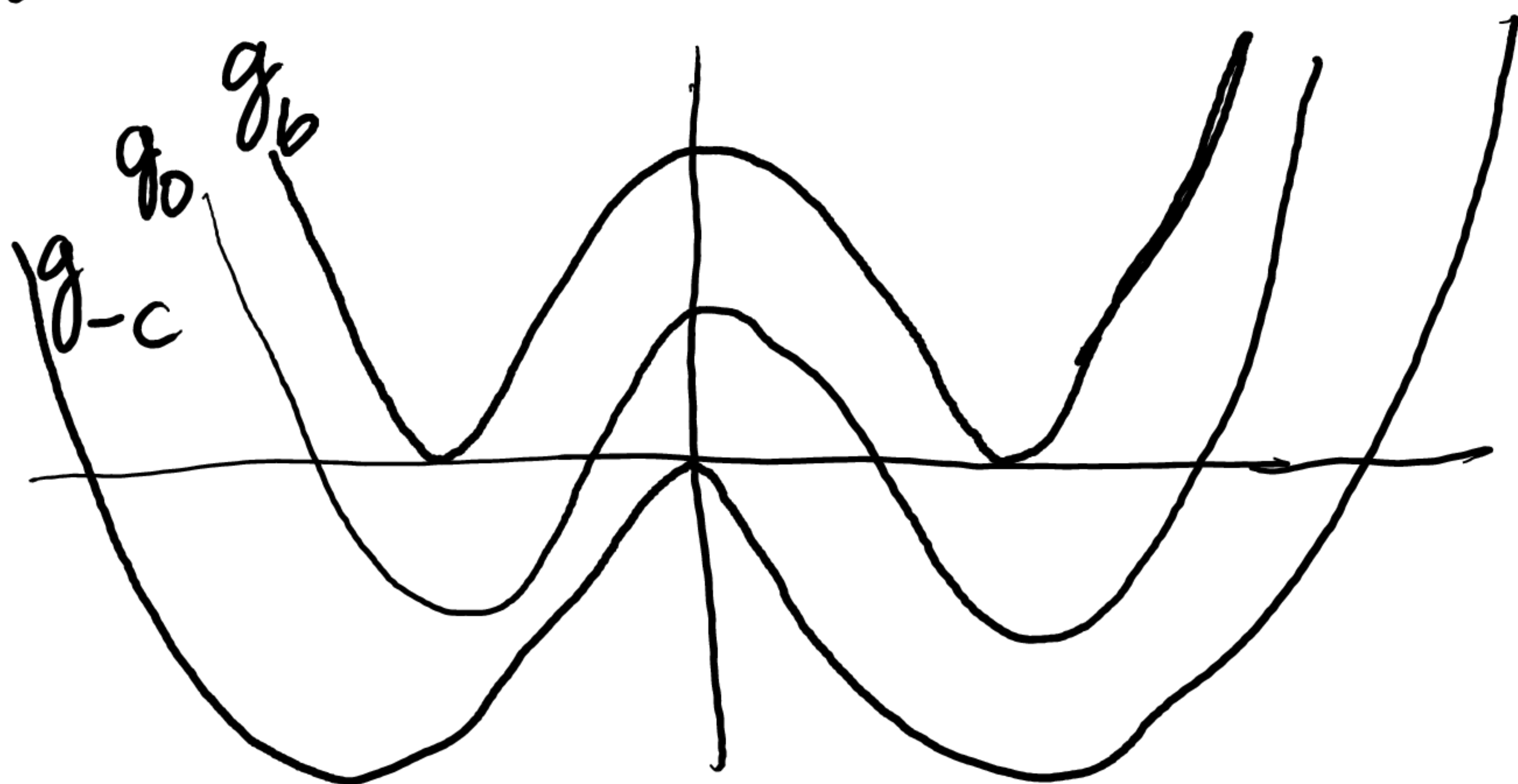


- Draw phase line for $x' = f(x)$
- Let $g_a(x) = f(x) + a$. Sketch bifurcation diagram for $x' = g_a(x)$ $a \in \mathbb{R}$
- Describe the different bifurcations that occur.

a) Add points $x_1^{+/-}$ & $x_2^{+/-}$



b) a represents a shift of f :



If $a > b$ then $g_a > 0$ so no EP
 $a > b$

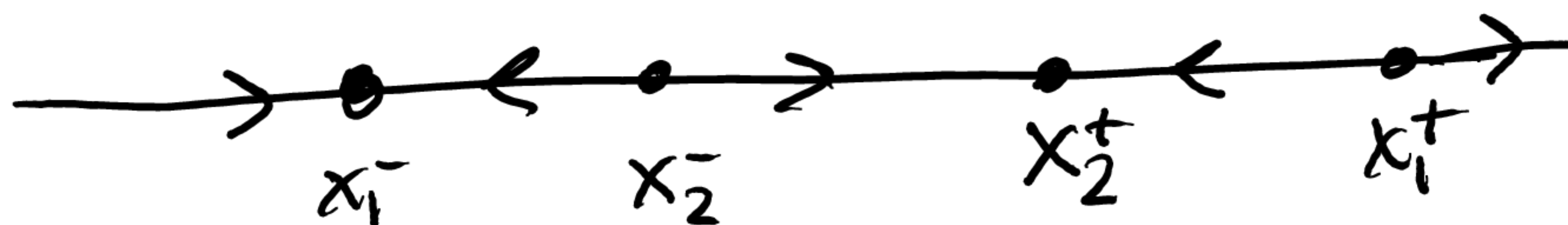
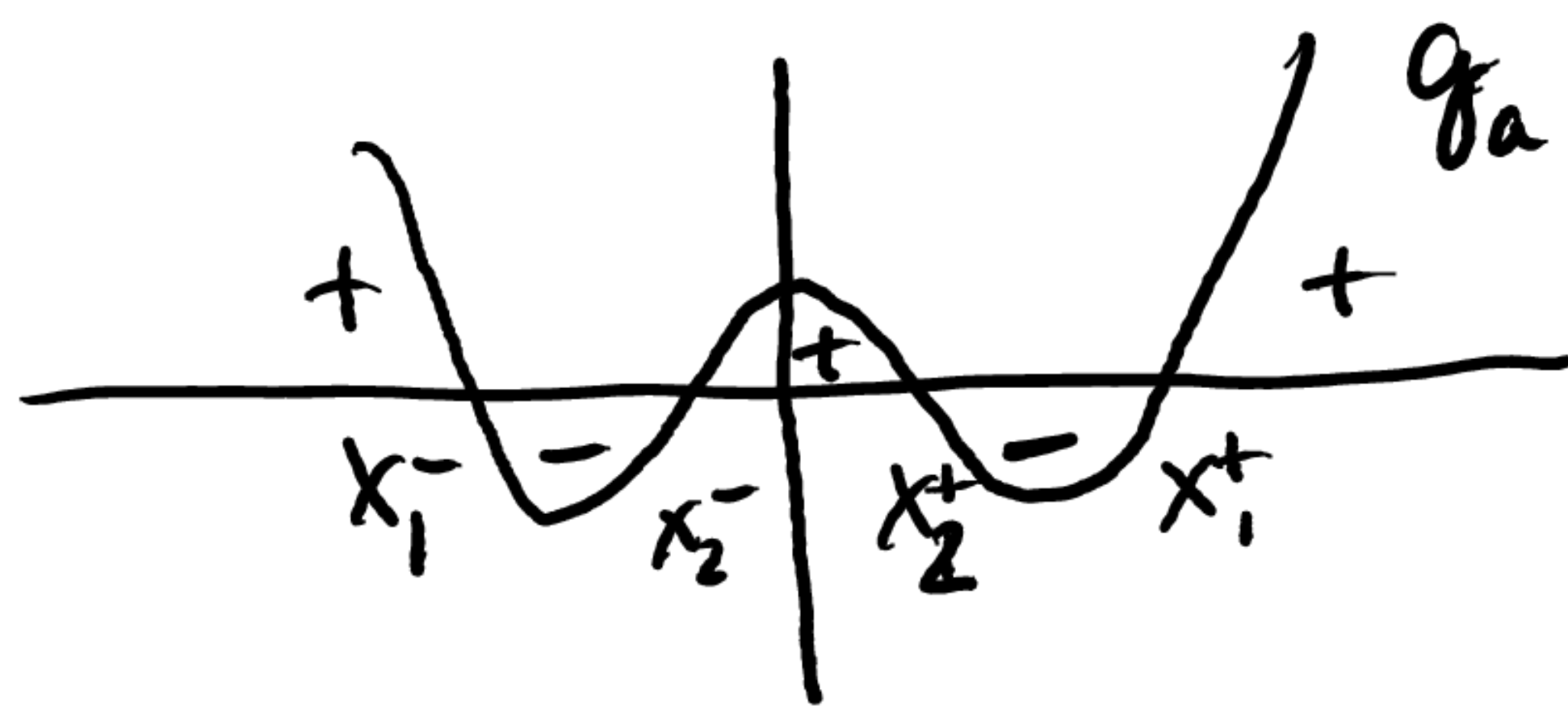


If $a = b$ then ODE has two EP

$$x_1^{+/-}(a) = x_2^{+/-}(a). \quad a = b$$

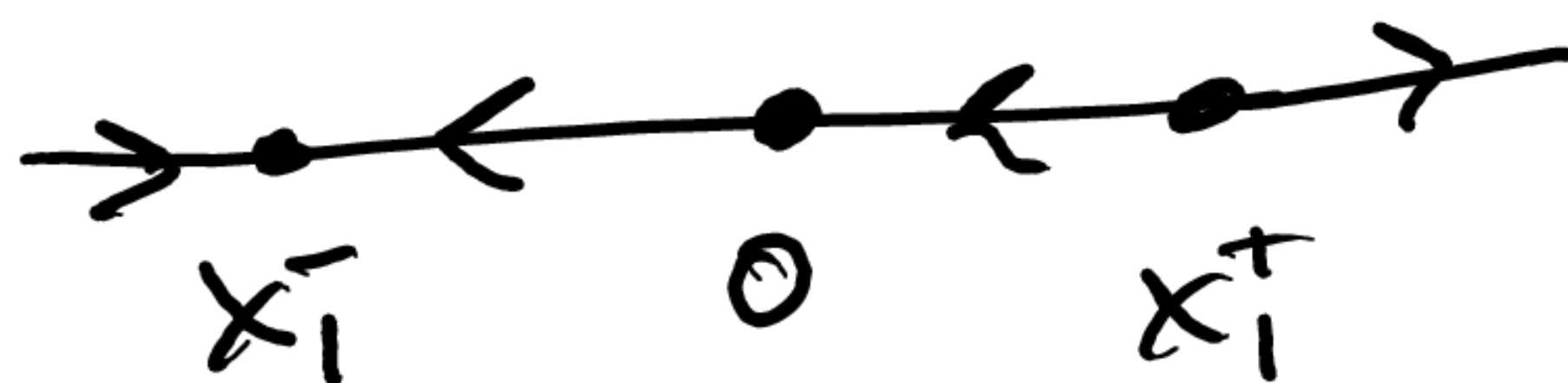


Let $f(0) =: c$. If $a \in (-c, b)$, then
four EP $x_1^{+/-}(a)$, $x_2^{+/-}(a)$



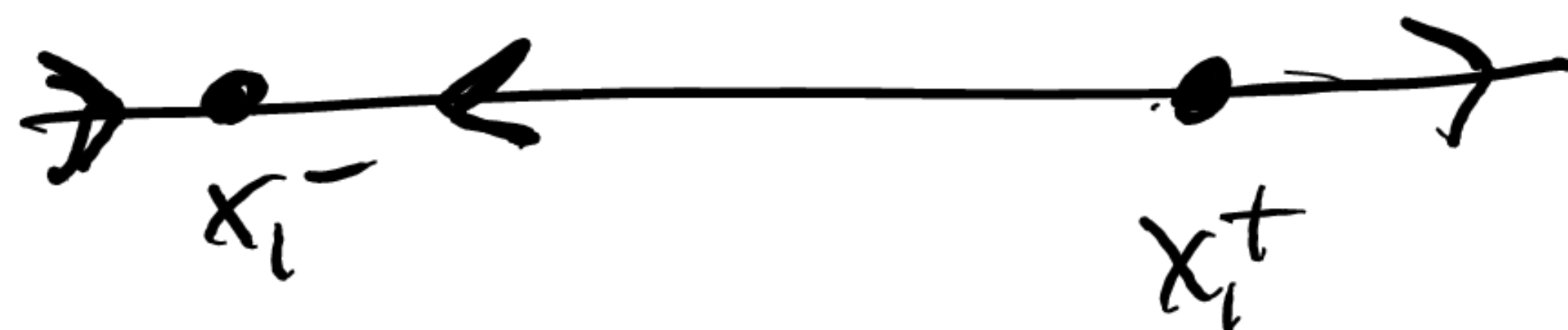
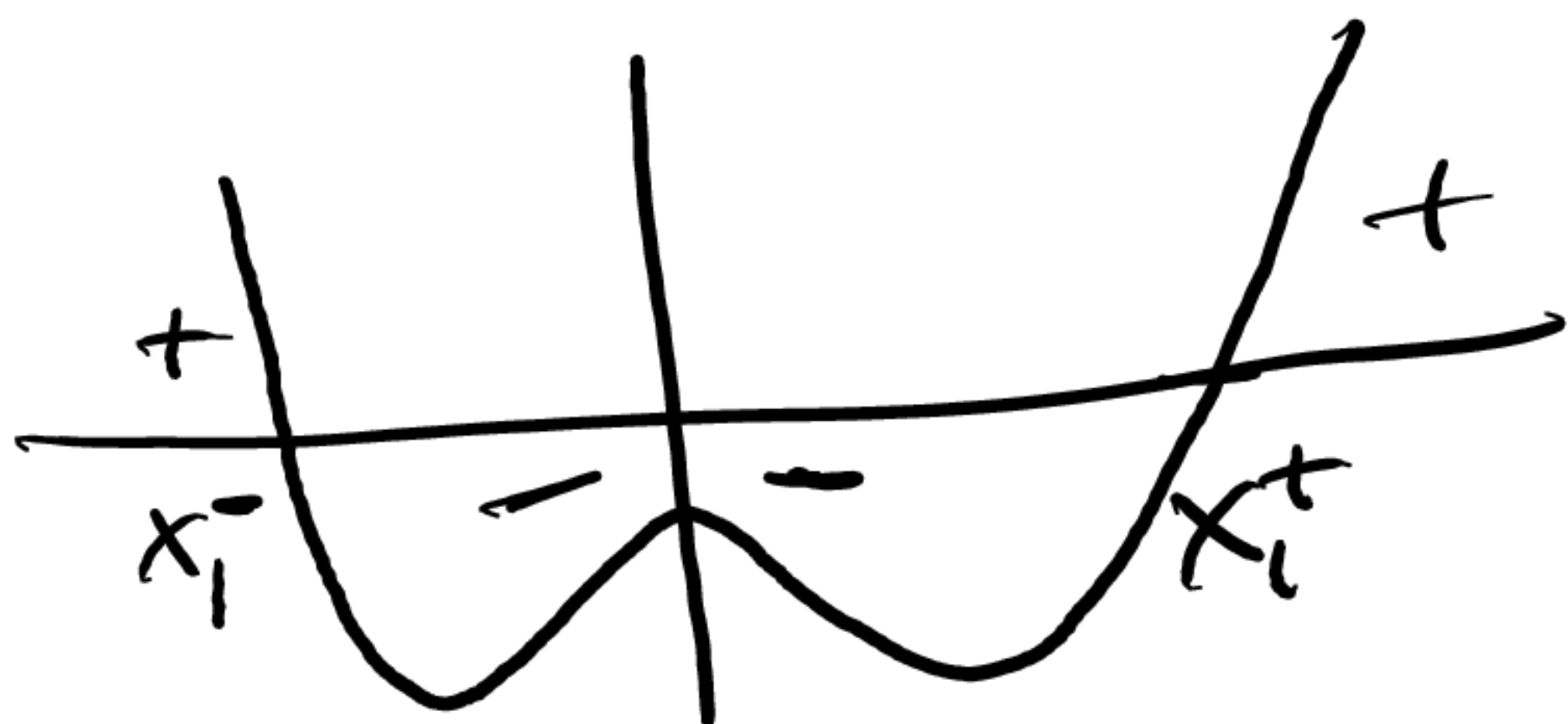
If $a = -c$, then 3 EP: $x_1^{+/-}(a)$ and

$$x_2^{+/-}(a) = 0$$

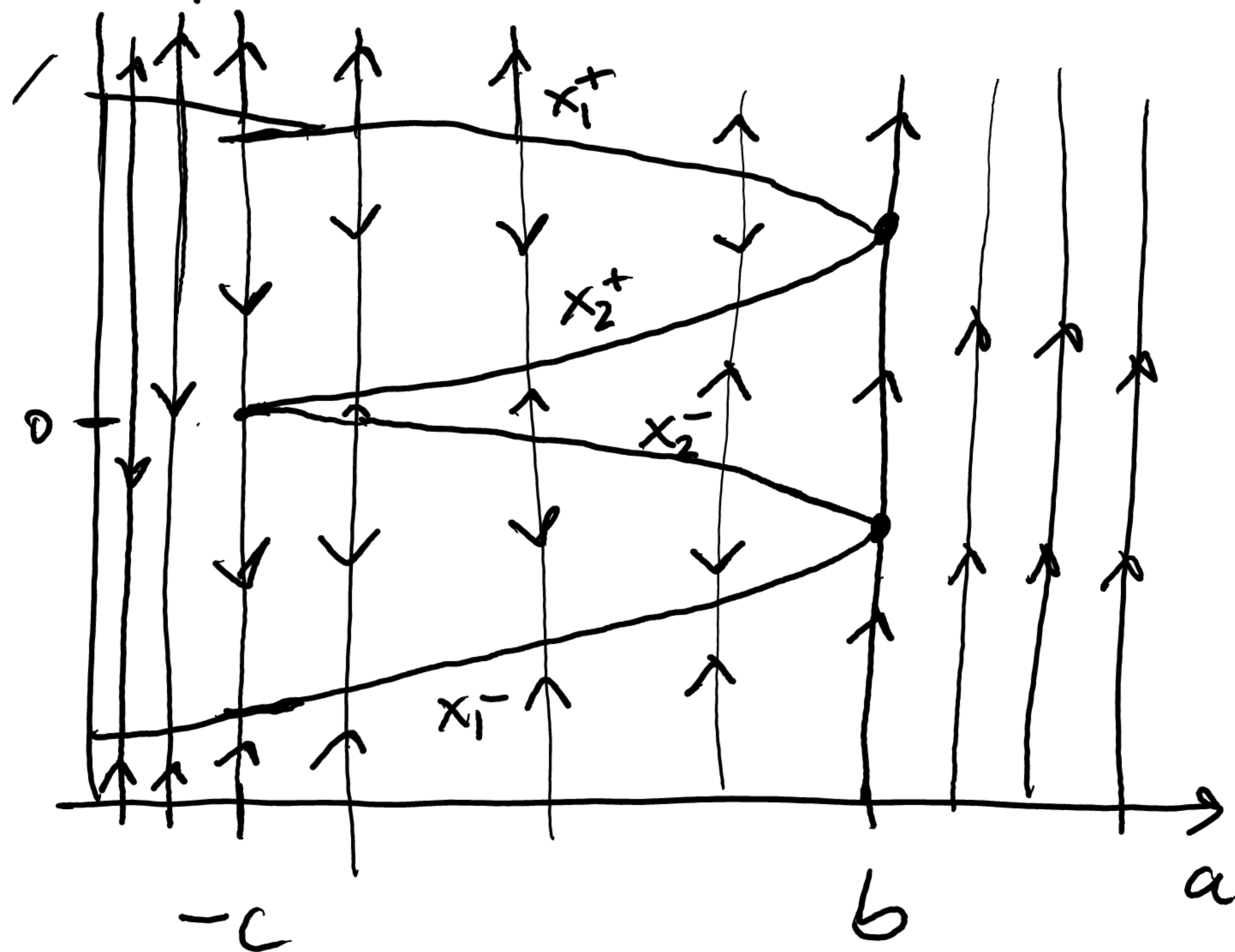


If $a < -c$, then ODE has two EP:

$$x_1^{+/-}(a)$$



Bifurcation diagram



c) $x_1^+(a)$ changes from source to neither at $a = b$.

same with x_2^-

$x_1^-(a)$ changes from sink to neither at $a = b$

same with x_2^+ .

6. (Simplified) Find a solution to

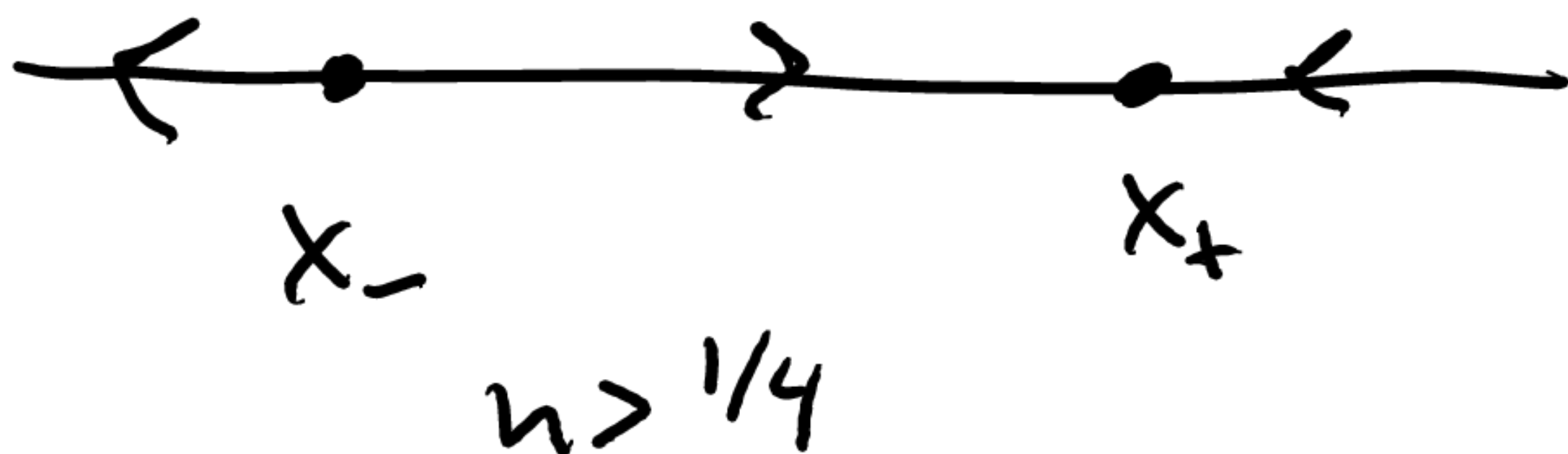
$$x' = x(1-x) - h \quad t \geq 0 \quad \text{for } h \in (0, \frac{1}{4})$$

and $x(0) = x_0 \in (x_-(h), x_+(h))$,

where $x_{+/-} = \frac{1 \pm \sqrt{1-4h}}{2}$

Sol: EP to ODE are $x_{+/-}(h)$.

Phase line
 $0 < h < \frac{1}{4}$



Know that $x(t)$ remains in
 (x_-, x_+) for t in some interval
 $[0, T)$.

By Sol (for $t \in [0, T)$)

$$\int \frac{1}{\underbrace{(x-x_-)(x-x_+)}} dx = -\int 1 dt$$

$$\int \frac{\frac{1}{x-x_-} - \frac{1}{x-x_+}}{x_- - x_+} dx = -\int 1 dt$$

$$\ln \left| \frac{x-x_-}{x-x_+} \right| = (x_+ - x_-)t + C$$

$$\frac{|x(t) - x_-|}{|x(t) - x_+|} = C \cdot \exp((x_+ - x_-)t)$$

$$x(t) \in (x_-, x_+) \Rightarrow \frac{x - x_-}{x_+ - x}$$

$$\frac{X(t) - X_-}{X_+ - X(t)} = C \cdot e^{(X_+ - X_-)t}$$

$$\Rightarrow X(t) = \frac{X_- + C X_+ e^{(X_+ - X_-)t}}{1 + C e^{(X_+ - X_-)t}}$$

$$X(0) = X_0 \Rightarrow C = \frac{X_0 - X_-}{X_+ - X_0} > 0$$

It holds that $\overbrace{X(t) \in (X_-, X_+)}^{\text{verify}} \quad \forall t \geq 0,$
 so this is a global solution.

7. Consider the non autonomous ODE

$$x' = f(x, t) := \begin{cases} x - 4 & \text{if } t < 5 \\ 2 - x & \text{if } t \geq 5 \end{cases}$$

a) Find a solution when $x(0) = 4$.

b) _____ || _____ $x(0) = 3$.

Describe the qualitative behavior of this solution

c) Describe qualitative behavior of any solution as $t \rightarrow \infty$.

Sol a) For $t < 5$, integrating factor
 $x' = x - 4 \mid e^{-t}$ yields

$$\frac{d}{dt}(xe^{-t}) = -4e^{-t} \mid \int dt$$

$$x(t) = 4 + ce^t$$

$$x(0) = 4 \Rightarrow c = 0 \text{ so } x(t) = 4 \quad t < 5$$

For $t \geq 5$

$$x' = 2 - x \quad | \quad e^t \quad (e^{at} \text{ with } a=-1)$$

yields $x(t) = 2 + Ce^{-t}$

$$x(5) = 4 \Rightarrow C = 2e^5$$

$$x(t) = \begin{cases} 4 & t \in [0, 5) \\ 2(1 + e^{5-t}) & t \geq 5 \end{cases}$$

b) If $x(0) = 3$, then

$$x(t) = 4 + Ce^t \quad t < 5 \quad \text{with}$$

$$C = -1$$

$$\text{so } x(5) = 4 - e^5 \quad \text{and since}$$

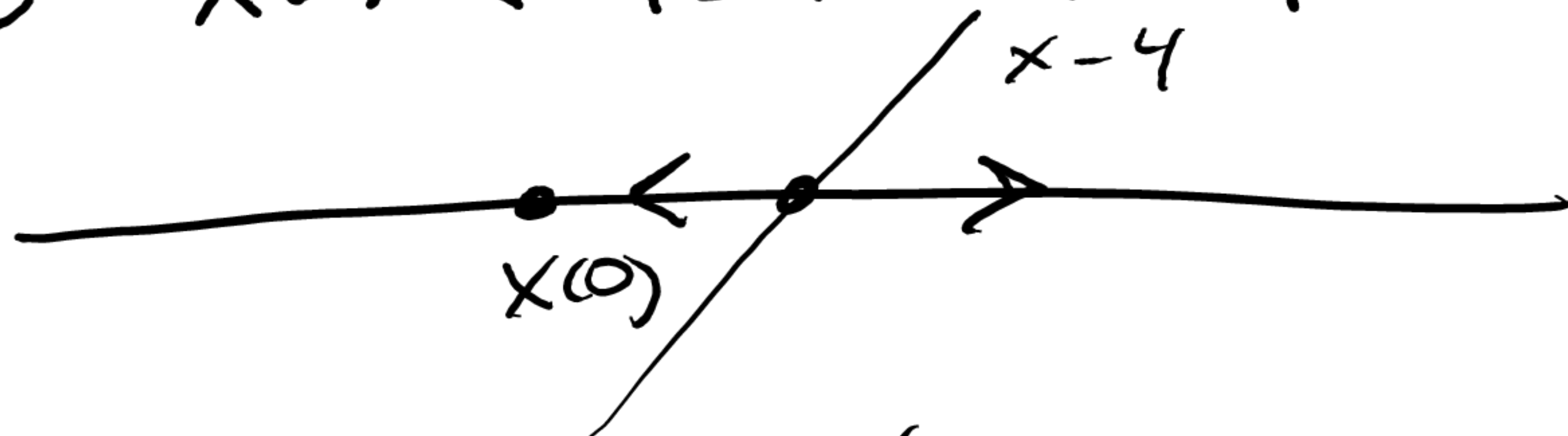
$$x(t) = 2 + Ce^{-t} \quad \text{for } t \geq 5$$

$$C = (2 - e^5)e^5$$

$$x(t) = \begin{cases} 4 - e^t, & t < 5 \\ 2 + (2 - e^5)e^{5-t} & t \geq 5 \end{cases}$$

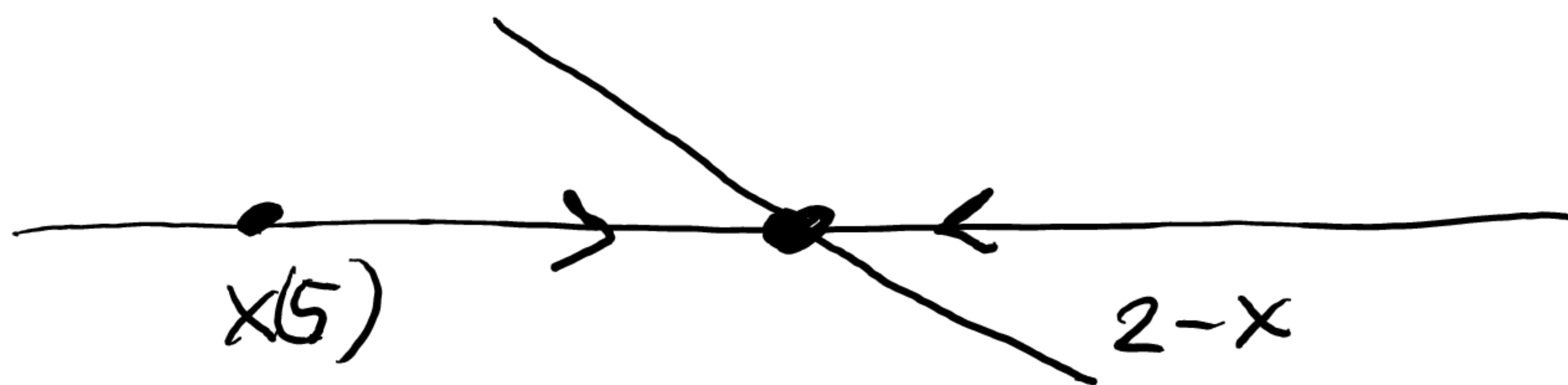
Qualitative behavior:

As $x(0) < 4 = x^* \text{ EP}$ for $t < 5$



$x'(t) < 0$ for $t < 5$

As $x(5) = 4 - e^5 < 2 = x^* \text{ EP}$
 $t \geq 5$



$x(t) \rightarrow x^*$ as $t \rightarrow \infty$.

c) For $t \geq 5$, $x^* = 2$ is only EP
of $x' = 2 - x$ and it is a global
sink

(seen from $x(t) = 2 + (e^{-t})$),

so $\lim_{t \rightarrow \infty} x(t, x_0) = 2 \quad \forall x_0 \in \mathbb{R}$.

9. Consider $x' = a(t)x$ Assume $a: \mathbb{R} \rightarrow \mathbb{R}$ is integrable

a) Find a solution formula

b) Prove that your formula is a general solution.

Sol a) Let $A(t) = \int_0^t a(s) ds$,
and note that $\frac{d}{dt} e^{-A(t)} = -A'(t) e^{-A(t)}$
 $= -a(t) e^{-A(t)}$.

Use $e^{-A(t)}$ as integrating factor:

$$\underbrace{(x' - a(t)x) e^{-A(t)}} = 0$$

$$\frac{d}{dt} (x e^{-A(t)}) = 0 \quad \Bigg| \quad \int dt$$

$$\Rightarrow \boxed{x(t) = C e^{A(t)} = C e^{\int_0^t a(s) ds}} \\ C \in \mathbb{R}$$

b) Follows from that any solution $y(t)$ must satisfy $\frac{d}{dt} (y e^{A(t)}) = 0$

and integrating, any solution must be
on form

$$y(t) = C e^{At}.$$

14. Consider $x' = P(t)x$ with

$P \in C^1(\mathbb{R})$ being T -periodic

Show that all solutions of this

eq. are periodic iff $\int_0^T P(s) ds = 0$

Sol By exercise 1.9, general solution of ODE is

$$x(t) = C e^{\int_0^t P(s) ds}.$$

Since $P(t+T) = P(t) \forall t$, we note that

$$\int_T^{T+t} P(s) ds = \int_0^t P(s+T) ds = \int_0^t P(s) ds \quad (1)$$

and

$$\begin{aligned} x(t+T) &= C e^{\int_0^{T+t} P(s) ds} \\ &= C e^{\int_0^T P(s) ds + \int_T^{T+t} P(s) ds} \\ &\stackrel{(1)}{=} C e^{\int_0^T P(s) ds + \int_0^t P(s) ds} \end{aligned}$$

We conclude that for a solution with $C \neq 0$,

$$X(T+t) = X(t) \iff \int_0^T P(s) ds = 0.$$