

Lecture 5 2/2

An n -dimensional first-order system of ODEs is given by

$$\left. \begin{aligned} x_1' &= f_1(x_1, \dots, x_n, t) \\ &\vdots \\ x_n' &= f_n(x_1, \dots, x_n, t) \end{aligned} \right\} \quad t \in I \quad (1)$$

where $x_j: I \rightarrow \mathbb{R}$ and $f_j: \mathbb{R}^n \times I \rightarrow \mathbb{R}$ are $C^\infty(\mathbb{R}^n \times I)$

Compact notation: Let $\underline{x}(t) := \begin{bmatrix} x_1(t) \\ \vdots \\ x_n(t) \end{bmatrix}$

$$\text{and } F(\underline{x}, t) = \begin{bmatrix} f_1(\underline{x}, t) \\ \vdots \\ f_n(\underline{x}, t) \end{bmatrix}.$$

Then (1) can be written

$$\underline{x}(t) = F(\underline{x}, t) \quad (2)$$

Convention Vectors, vector-val fns $\underline{X}, F(\underline{X})$
scalars, scalar-val fns $x, f(x)$

Def The ODE (1)/(2) is
called autonomous if $F = F(\underline{X})$
(That is, if $f_j(\underline{X}, t) = f_j(\underline{X}) \forall j$.)

Def For an autonomous n -dim system
 $\underline{X}' = F(\underline{X})$, $\underline{X}^* \in \mathbb{R}^n$ is
called an EP if $F(\underline{X}^*) = 0$ ($\in \mathbb{R}^n$).

Special case $n=2$

$$x' = f_1(x, y, t)$$

$$y' = f_2(x, y, t)$$

are called planar systems ($n=2$).

Ex

$$x' = y$$

$$y' = -x$$

Note A solution path $\Sigma(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$
for $t \in I$ satisfies

$$\Sigma'(t) = F(\Sigma(t)) = \begin{bmatrix} y(t) \\ -x(t) \end{bmatrix} \quad (3)$$

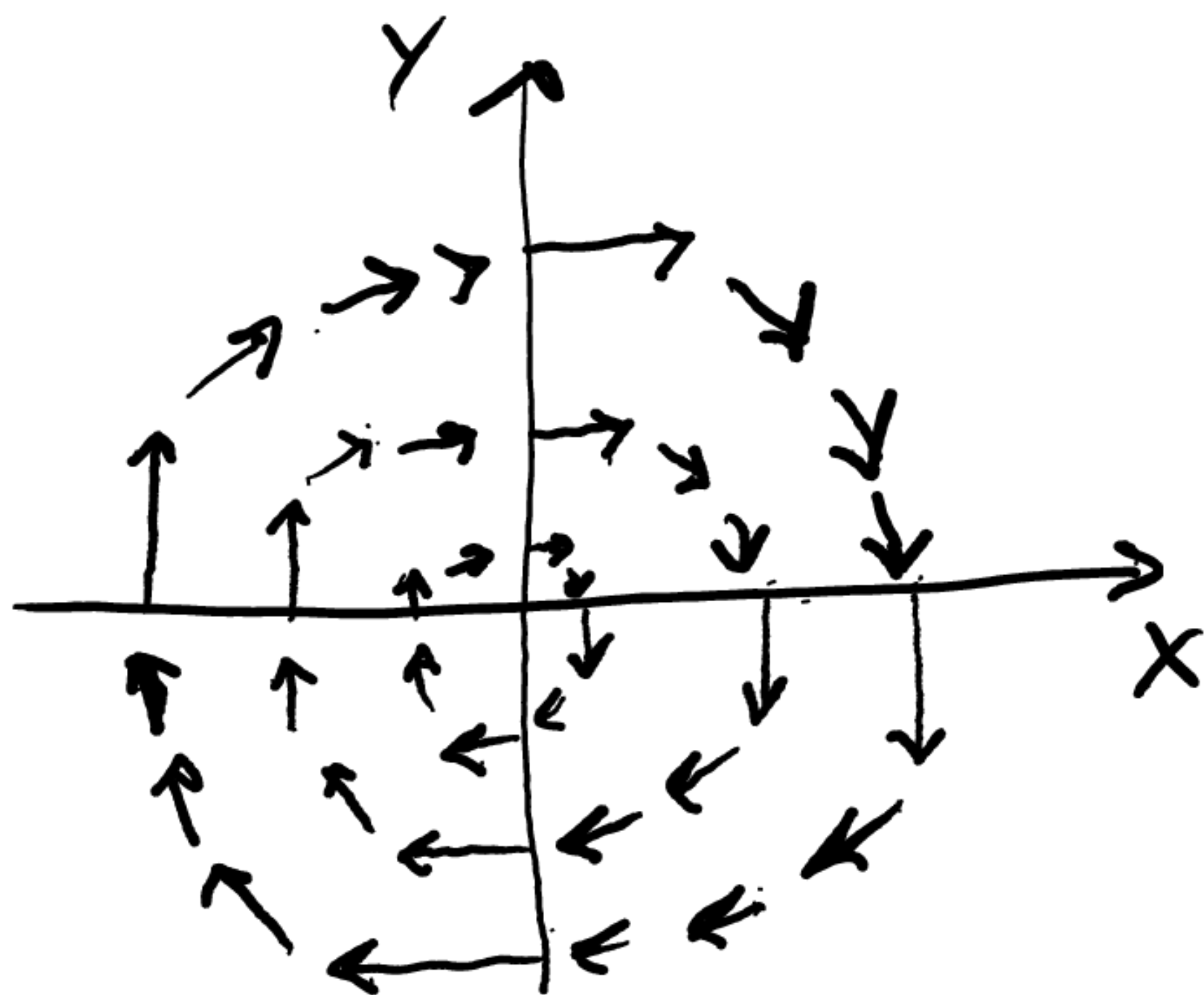
Note every solution trajectory $\{\Sigma(s)\}_{s \in I}$
that goes through a point $\begin{bmatrix} x \\ y \end{bmatrix} \in \mathbb{R}^2$
say at time t (meaning $\Sigma(t) = \begin{bmatrix} x \\ y \end{bmatrix}$),
must satisfy

$$\Sigma'(t) = F(\Sigma(t)) = \begin{bmatrix} y \\ -x \end{bmatrix}$$

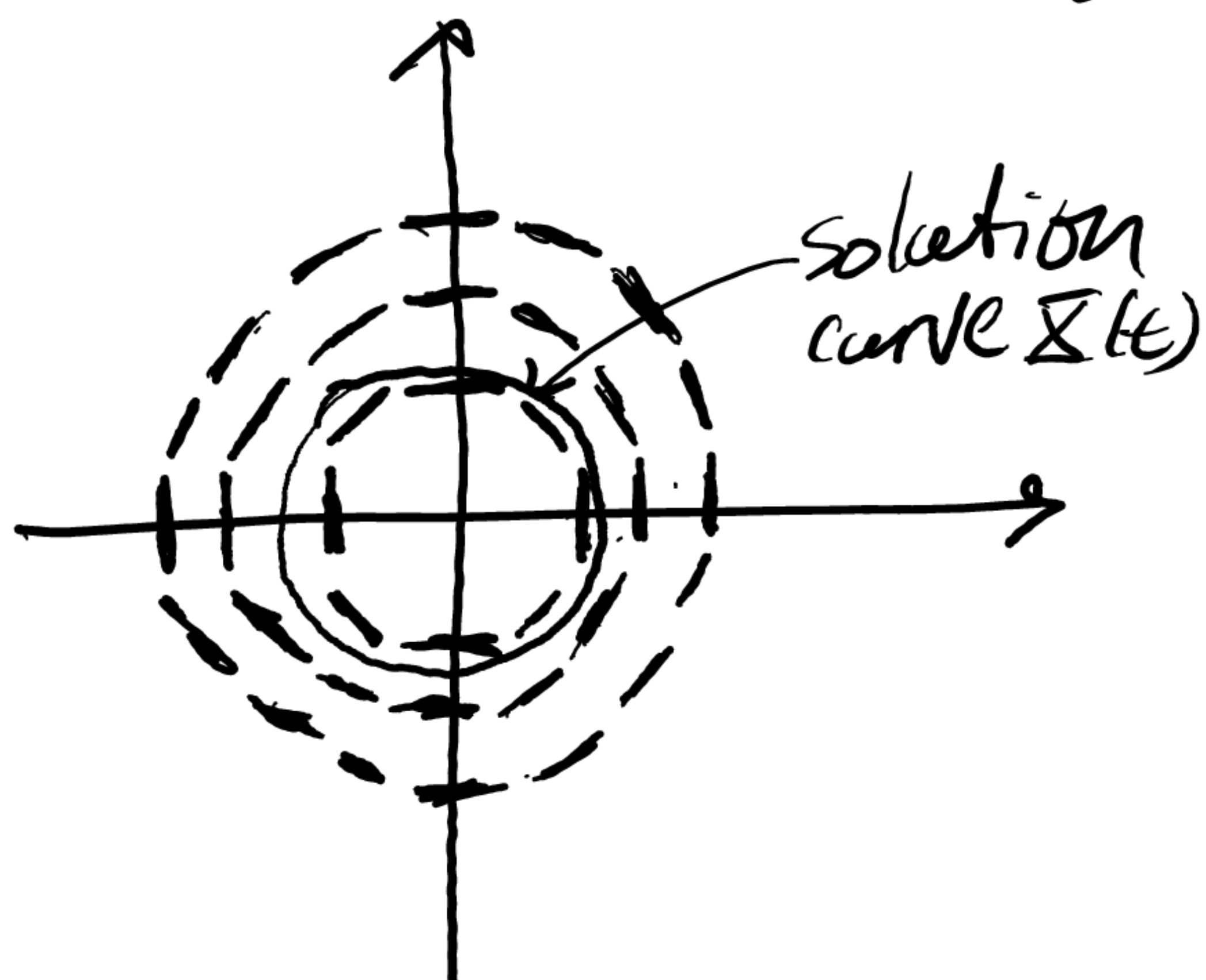
Solution trajectories are
tangent to the vector field in
every point they go through.

Plots of $F\begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} y \\ -x \end{bmatrix}$

Vector field F



Direction field (scaled F)



For every $r \geq 0$, $\underline{X}(t) = r \begin{bmatrix} \sin(t) \\ \cos(t) \end{bmatrix}$

is a solution, as

$$\underline{X}'(t) = r \begin{bmatrix} \frac{d}{dt} \sin(t) \\ \frac{d}{dt} \cos(t) \end{bmatrix} = \begin{bmatrix} x_2(t) \\ -x_1(t) \end{bmatrix}$$

First-order systems of ODE also relate to scalar-valued ODE of order $n \geq 2$:

$$X^{(n)} = f(X, X', X^{(2)}, \dots, X^{(n)}, t) \quad (4)$$

where $X^{(n)} = \frac{d^n}{dt^n} X(t)$.

$$\left. \begin{array}{l} \text{Let } X_1 := X \\ X_2 := X' \\ \vdots \\ X_{n-1} := X^{(n-2)} \\ X_n := X^{(n-1)} \end{array} \right\} \Rightarrow \begin{array}{l} X_1' = X' = X_2 \\ X_2' = X_3 \\ \vdots \\ X_{n-1}' = X_n \\ X_n' = X^{(n)} = f(X_1, \dots, X_n, t) \end{array}$$

i.e.

$$\mathbf{X}'(t) = \begin{bmatrix} X_2 \\ \vdots \\ X_n \\ f(\mathbf{X}, t) \end{bmatrix} =: \mathbf{F}(\mathbf{X}, t)$$

(X, t) of system is solution to (4).

Ex

Free falling object

$$ma = F = mg$$
$$\& a = y''$$

$$y''(t) = g (= -9.81)$$

$(\gamma_1, \gamma_2) := (\gamma, \gamma')$. Then

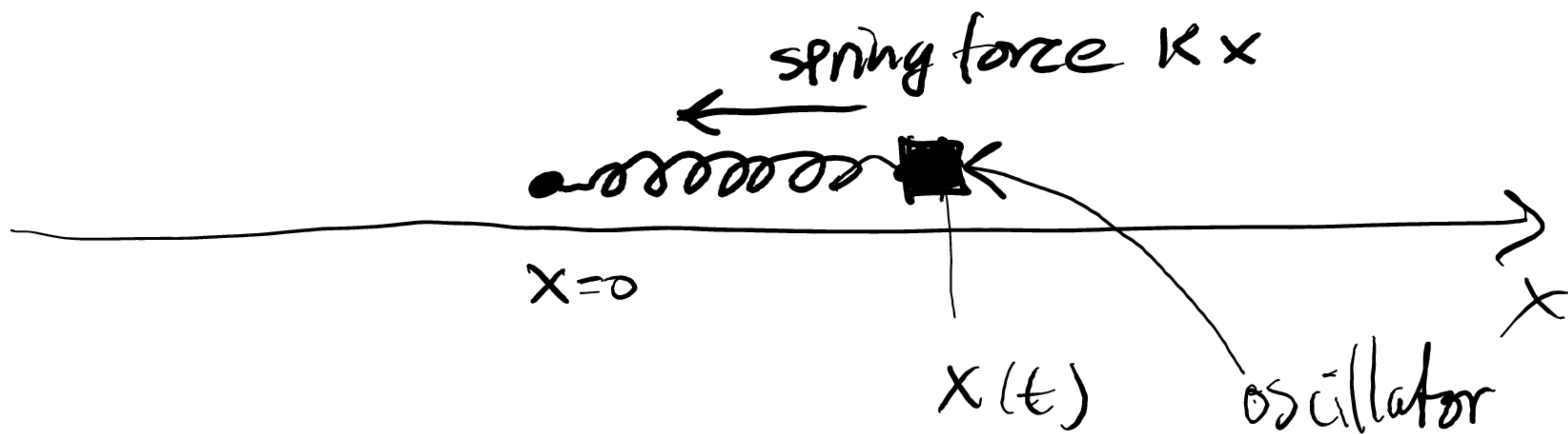
$$\mathcal{Y}' = \begin{bmatrix} \gamma_1' \\ \gamma_2' \end{bmatrix} = \begin{bmatrix} \gamma_2 \\ g \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \mathcal{Y} + \begin{bmatrix} 0 \\ g \end{bmatrix}$$

(linear, inhomogeneous ODE).

Ex Damped Harmonic oscillator

$$\underbrace{m x''}_{\text{forces on oscillator}} = \underbrace{-kx}_{\text{spring force}} - \underbrace{b x'}_{\text{damping}}, \text{ with } k > 0$$
$$b \geq 0$$

$x(t)$ = displacement of oscillator from $x=0$



System: $x_1 = x \Rightarrow x_1' = x_2$
 $x_2 = x' \Rightarrow x_2' = -\frac{kx}{m} - \frac{bx'}{m}$
 $= -\frac{kx_1}{m} - \frac{b}{m}x_2$

so $\bar{x}' = \begin{bmatrix} 0 & 1 \\ -\frac{k}{m} & -\frac{b}{m} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = A\bar{x}$

Planar linear, homogeneous systems

We now study the $n=2$ ODE

$$\bar{x}' = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \quad (5)$$

Compact notation: $\bar{x}' = A\bar{x} =: F(\bar{x})$

$\bar{x}^* \in \mathbb{R}^2$ is an EP if $F(\bar{x}^*) = A\bar{x}^* = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$

Proposition 1 (Equilibrium solutions)

Consider ODE (5) with $A \neq 0$.

If $\det(A) \neq 0$, then ODE has a unique EP: $\bar{x}^* = A^{-1} \begin{bmatrix} 0 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$.

If $\det(A) = 0$, then

$A \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix} = 0$ gives a line of EP

$$\begin{pmatrix} x \\ cx \end{pmatrix}_{x \in \mathbb{R}} \quad \text{where } c = -\frac{a_{11}}{a_{12}} \text{ and/or } -\frac{a_{21}}{a_{22}}$$

$$\left(\begin{array}{l} a_{11}x_1^* + a_{12}x_2^* = 0 \\ a_{21}x_1^* + a_{22}x_2^* = 0 \end{array} \right),$$

Ex $A = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$. Then

$$A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \Leftrightarrow x_2 = -x_1$$

$$\text{EP of } \bar{x}' = A\bar{x} : \begin{pmatrix} x \\ -x \end{pmatrix}_{x \in \mathbb{R}}.$$

Proposition 2 (Real eigenvector solutions)

Suppose $A \in \mathbb{R}^{2 \times 2}$ has
 (λ, V) with $\lambda \in \mathbb{R}$ and $V \in \mathbb{R}^2$. Then
 $\mathcal{I}(t) := e^{\lambda t} V$ solves $\mathcal{I}' = A\mathcal{I}$.

Proof $\mathcal{I}'(t) = e^{\lambda t} \lambda V = e^{\lambda t} A V$
 $= A(e^{\lambda t} V) = A\mathcal{I}$ $\square$

Asymptotic properties:

Let $|\mathcal{I}| := \sqrt{y_1^2 + y_2^2}$.

• If $\lambda > 0$, then

$$\lim_{t \rightarrow \infty} |\mathcal{I}(t)| = \infty \quad \& \quad \lim_{t \rightarrow -\infty} \mathcal{I}(t) = 0$$

• If $\lambda < 0$, then

$$\lim_{t \rightarrow \infty} \mathcal{I}(t) = 0 \quad \text{and} \quad \lim_{t \rightarrow -\infty} |\mathcal{I}(t)| = \infty$$

Proposition 3 (superposition principle)

Suppose $\underline{x}_1$ and $\underline{x}_2$ both solve $\underline{x}' = A\underline{x}$. Then

$\underline{y} = \alpha \underline{x}_1 + \beta \underline{x}_2$ for any $\alpha, \beta \in \mathbb{R}$ also solves the ODE.

Proof

$$\begin{aligned}\underline{y}' &= \alpha \underline{x}_1' + \beta \underline{x}_2' \\ &= \alpha A\underline{x}_1 + \beta A\underline{x}_2 \\ &= A(\underbrace{\alpha \underline{x}_1 + \beta \underline{x}_2}_{\underline{x}})\end{aligned}$$



Theorem If $A \in \mathbb{R}^{2 \times 2}$ has real eigenvalues λ_1, λ_2 with linearly independent eigenvectors $V_1, V_2 \in \mathbb{R}^2$, then
$$\underline{X} = \alpha e^{\lambda_1 t} V_1 + \beta e^{\lambda_2 t} V_2 \quad \alpha, \beta \in \mathbb{R}$$
 is the general solution of
$$\underline{X}' = A \underline{X}.$$

Proof Prop 2 $\Rightarrow \underline{X}_i(t) = e^{\lambda_i t} V_i$ for $i=1,2$ are both solutions.

Prop 3 $\Rightarrow \underline{X}$ is a solution for any α, β .

General solution: Suppose $Z(t)$ is a solution. Since $\text{span}(V_1, V_2) = \mathbb{R}^2$, we

can write
$$Z(t) = \mu_1(t) V_1 + \mu_2(t) V_2$$

for some $\mu_1, \mu_2 \in C^1(\mathbb{R})$.

And $Z' = AZ$ implies that

$$\begin{aligned}\mu_1'(t)V_1 + \mu_2'(t)V_2 &= \mu_1(t)AV_1 + \mu_2(t)AV_2 \\ &= \lambda_1\mu_1(t)V_1 + \lambda_2\mu_2(t)V_2\end{aligned}$$

$$\Rightarrow (\lambda_1\mu_1(t) - \mu_1'(t))V_1 + (\lambda_2\mu_2(t) - \mu_2'(t))V_2 = 0$$

for any $t \in \mathbb{R}$.

Linear independence implies

$$\mu_1' = \lambda_1\mu_1, \quad \mu_2' = \lambda_2\mu_2 \quad \forall t$$

with general solutions

$$\mu_1(t) = \alpha e^{\lambda_1 t}, \quad \mu_2(t) = \beta e^{\lambda_2 t}, \quad \alpha, \beta \in \mathbb{R}.$$

Hence, any solution

$$Z(t) = \mu_1(t)V_1 + \mu_2(t)V_2 = \alpha e^{\lambda_1 t}V_1 + \beta e^{\lambda_2 t}V_2$$

Recall: If $A \in \mathbb{R}^{2 \times 2}$ has real distinct eigenvalues $\lambda_1 \neq \lambda_2$, then its eigenvectors are linearly independent so the previous theorem applies.

Corollary If $A \in \mathbb{R}^{2 \times 2}$ has real eigenvalues and lin. independent eigenvectors, then

$$\mathbf{x}' = A\mathbf{x}$$

$$\mathbf{x}(0) = \mathbf{x}_0$$

has a unique solution.

Proof General solution of $\mathbf{x}' = A\mathbf{x} =$

$$\mathbf{x}(t) = \alpha e^{\lambda_1 t} \mathbf{v}_1 + \beta e^{\lambda_2 t} \mathbf{v}_2$$

with $\mathbf{v}_1, \mathbf{v}_2$ linearly indep.

Initial condition

$$\alpha \mathbf{v}_1 + \beta \mathbf{v}_2 = \mathbf{x}(0) = \mathbf{x}_0$$

uniquely determines α, β .



$$\underline{Ex} \quad \bar{X} = \underbrace{\begin{bmatrix} 1 & 3 \\ 1 & -1 \end{bmatrix}}_A \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

Char eq. $\det \begin{pmatrix} 1-\lambda & 3 \\ 1 & -1-\lambda \end{pmatrix} = 0$

$$-(1-\lambda)(1+\lambda) - 3 = 0$$

$$\lambda_1 = -2, \quad \lambda_2 = 2.$$

Eigenvectors:

$$(A - \lambda_i I) V_i = 0 \quad \text{gives solutions}$$

$$V_1 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 3 \\ 1 \end{bmatrix}.$$

General solution:

$$\bar{X}(t) = \alpha e^{-2t} \begin{bmatrix} 1 \\ -1 \end{bmatrix} + \beta e^{2t} \begin{bmatrix} 3 \\ 1 \end{bmatrix} = \begin{bmatrix} \alpha e^{-2t} + 3\beta e^{2t} \\ -\alpha e^{-2t} + \beta e^{2t} \end{bmatrix}$$