

Lecture 6, MAT3440 4/2

We consider $\dot{X} = AX$ for $A \in \mathbb{R}^{2 \times 2}$.

Recall, if A has eigenpairs (λ_i, v_i) $i=1,2$ with $\lambda_1, \lambda_2 \in \mathbb{R}$ and $v_1, v_2 \in \mathbb{R}^2$ linearly indep., then

$$X(t) = \alpha e^{\lambda_1 t} v_1 + \beta e^{\lambda_2 t} v_2 \quad \alpha, \beta \in \mathbb{R}$$

is the general solution.

And the EP of the ODE are the $X^* \in \mathbb{R}^2$ s.t. $AX^* = 0$. $X^* = 0$ is

the only EP when A is invertible.

Q: Properties of EP $\lambda^* = 0$?

How to solve ODE for general A ?

Def

Any mapping $B = P^{-1}AP$ where $P \in \mathbb{R}^{2 \times 2}$ is invertible is called a similarity transformation of A .

A and B are called similar.

Important property: B and A share the same eigenvalues

(as $Av = \lambda v \iff B(P^{-1}v) = \lambda(P^{-1}v)$).

Lemma If A and $B = P^{-1}AP$
and $\mathcal{Y}(t)$ solves $\mathcal{Y}' = B\mathcal{Y}$,
then $\mathcal{X}(t) := P\mathcal{Y}(t)$ solves $\mathcal{X}' = A\mathcal{X}$.

Proof $\mathcal{X}'(t) = P\mathcal{Y}'(t)$
 $= PB\mathcal{Y}(t) = AP\mathcal{Y}(t)$
 $= A\mathcal{X}(t)$



Algorithm for solving $\mathcal{X}' = A\mathcal{X}$:

1. Find P s.t. $B = P^{-1}AP$ is diagonal (or in Jordan form).
2. Solve $\mathcal{Y}' = B\mathcal{Y}$.
3. Solution $\mathcal{X} = P\mathcal{Y}$.

There are three types of $A \in \mathbb{R}^{2 \times 2}$:

① A has real eigenvalues and linearly independent eigenvectors

② A has one repeated real eigenvalue λ , and $\dim(\ker(A - \lambda I)) = 1$, meaning only one linearly indep eigenvector.

③ A has a pair of complex eigenvalues $\lambda_1 = a + ib$, $\lambda_2 = a - ib$.

Case II

Let (λ_i, v_i) $i=1,2$, denote eigenpairs with v_1, v_2 linearly independent.

Diagonalization: Take $P = [v_1 \ v_2]$. Then

$$\begin{aligned} B &= P^{-1} A P = P^{-1} [A v_1 \ A v_2] = P^{-1} [\lambda_1 v_1 \ \lambda_2 v_2] \\ &= [\lambda_1 P^{-1} v_1 \ \lambda_2 P^{-1} v_2] = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \text{ is diagonal,} \end{aligned}$$

$$\begin{bmatrix} y_1' \\ y_2' \end{bmatrix} = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} \quad \text{amounts to solving}$$

$$\left. \begin{array}{l} y_1' = \lambda_1 y_1 \\ y_2' = \lambda_2 y_2 \end{array} \right\} \Rightarrow \bar{Y}(t) = \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{bmatrix}$$

$$\text{and } \underline{X}(t) = P \bar{Y}(t) = c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2$$

Stability of EP $\underline{X}^* = \bar{Y}^* = 0$

If $\lambda_2 < \lambda_1$, then (for any c_1, c_2)

$$\lim_{t \rightarrow \infty} |\bar{Y}(t)| = \lim_{t \rightarrow \infty} |\underline{X}(t)| = 0,$$

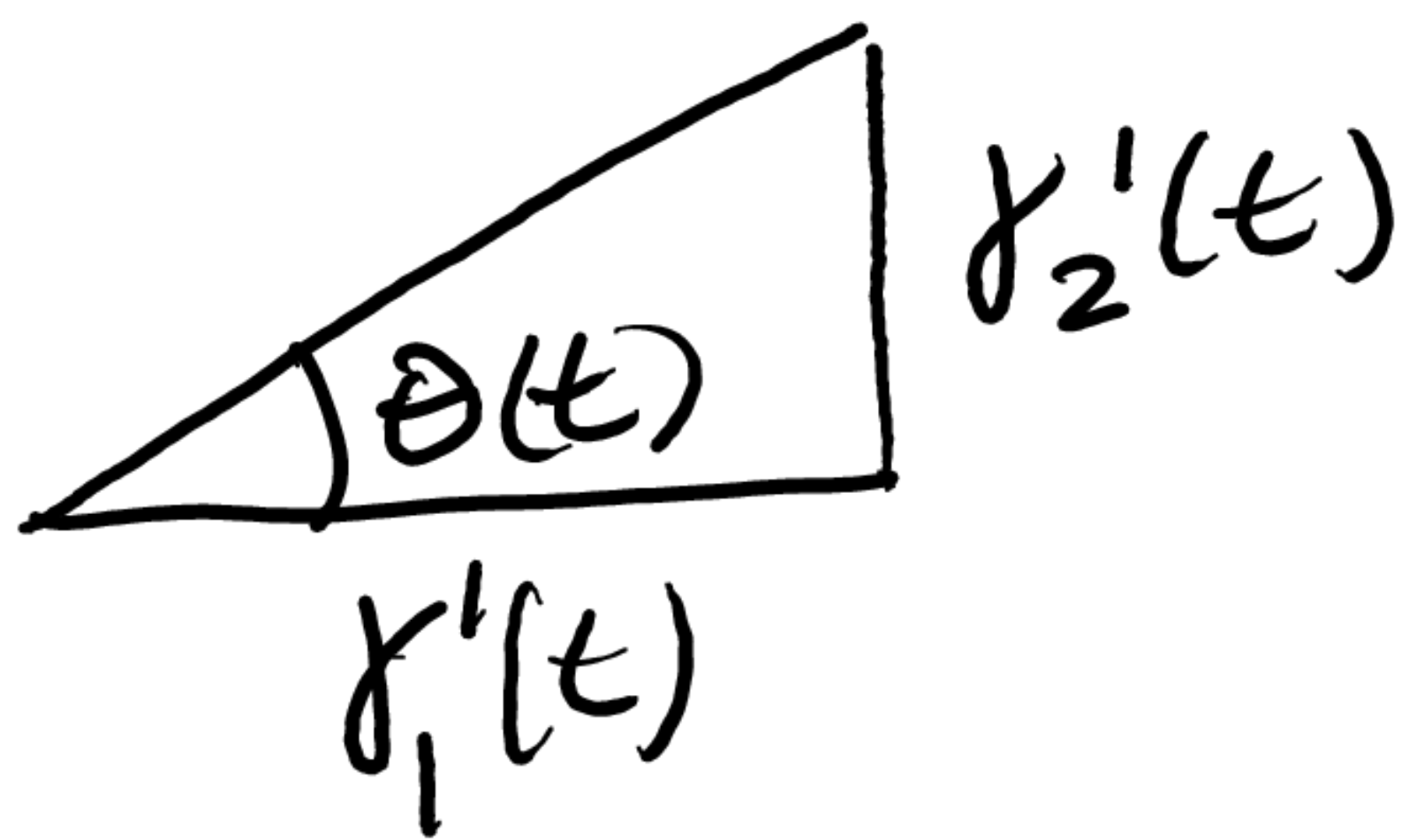
so $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$ is a sink for $\underline{X}(t)$ & $\bar{Y}(t)$.

Note Tangent of solution curve

$$y(t) = (y_1(t), y_2(t))$$

$$y'(t) = (c_1 \lambda_1 e^{\lambda_1 t}, c_2 \lambda_2 e^{\lambda_2 t})$$

so e.g. for $c_1, c_2 < 0$



and $\theta(t) \rightarrow 0$ as $t \rightarrow \infty$

$$\text{as } \frac{y_2'(t)}{y_1'(t)} = \frac{c_2 \lambda_2 e^{\lambda_2 t}}{c_1 \lambda_1 e^{\lambda_1 t}} \xrightarrow{\lambda_2 < \lambda_1} 0,$$

This means $\mathbf{Y}(t)$ tend to

0 in the $E_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ -direction (associated to weakest eigenvalue)

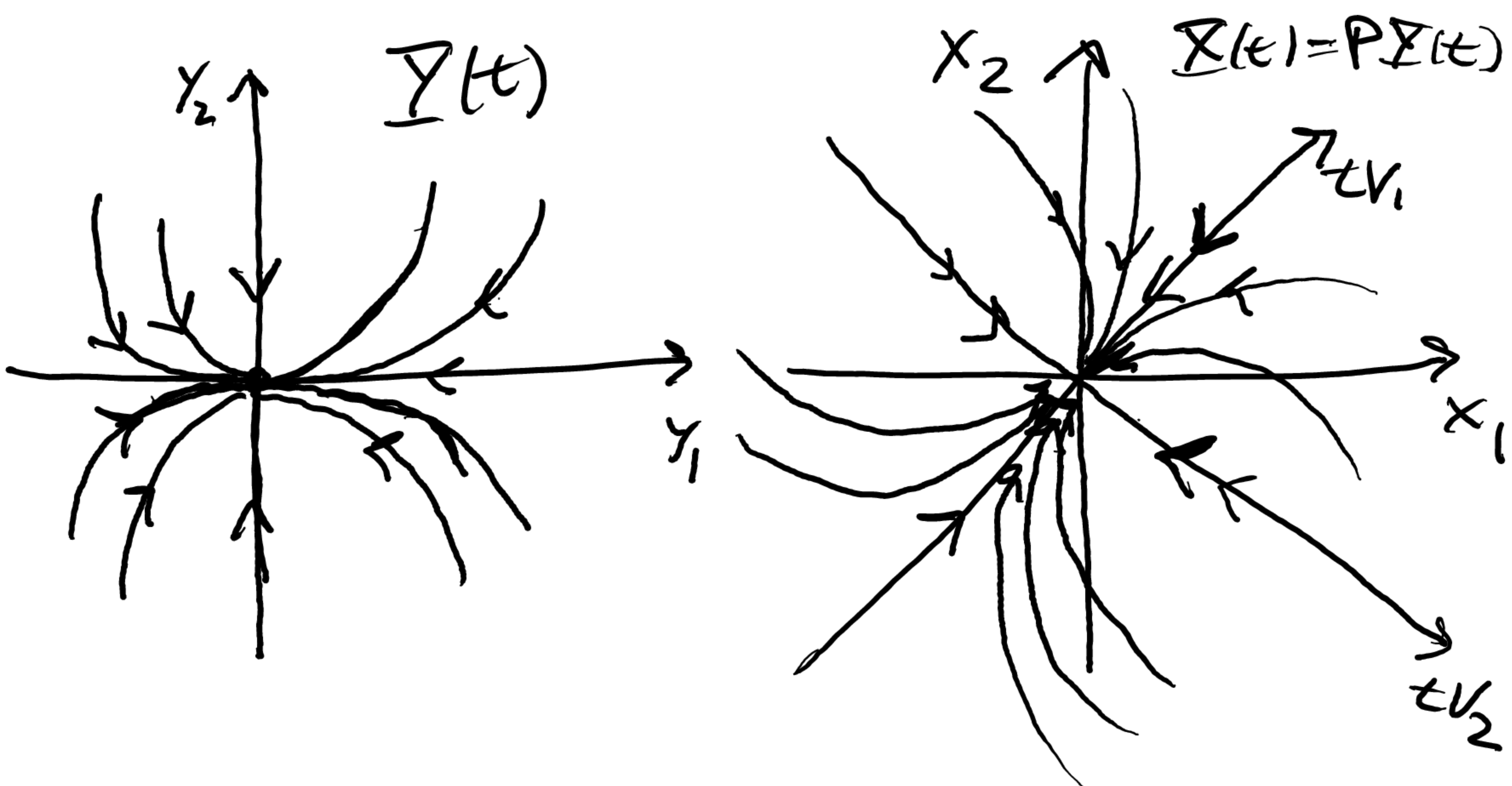
(If $\lambda_1 < \lambda_2 < 0$, it would be along y_2 -axis.)

$$\text{And } \mathbf{X}(t) = c_1 e^{\lambda_1 t} \mathbf{V}_1 + c_2 e^{\lambda_2 t} \mathbf{V}_2$$

tend to 0 in the $\mathbf{V}_1$ -direction

(along direction of weakest = smallest magnitude eigenvalue).

Phase portraits (supposing $V_1 = \begin{bmatrix} 1 \\ 1 \end{bmatrix}$, $V_2 = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$)



If $\lambda_1 < 0 < \lambda_2$ then

$$Z(t) = \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{bmatrix} \quad \text{and}$$

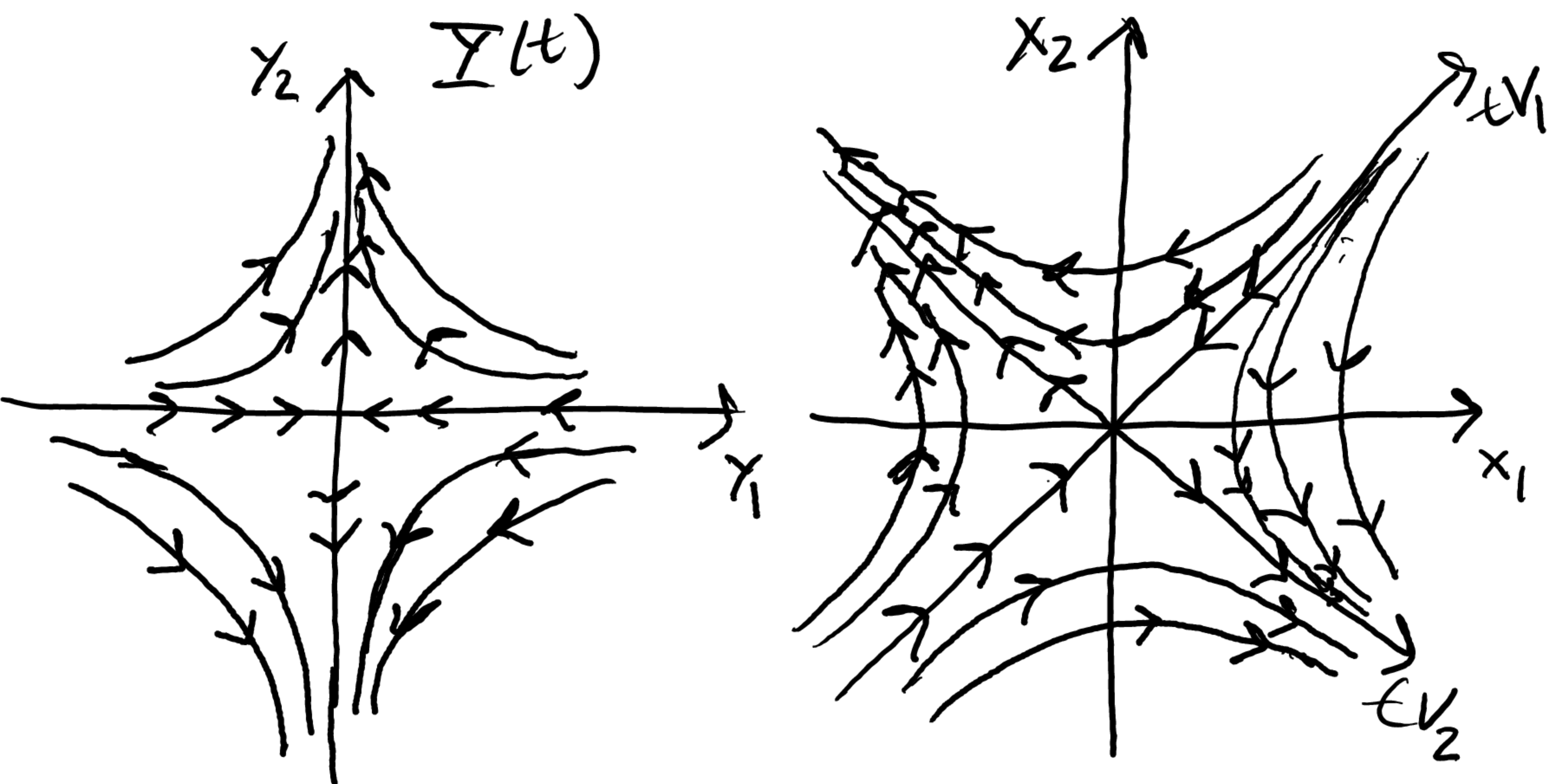
0 is a saddle point:

attractive in $E = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ direction, repulsive
in $E_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ direction near 0.

$$X(t) = c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2 \quad \text{is}$$

is attractive in V_1 -direction,
repulsive in V_2 -direction.

Phase portrait (V_1, V_2 as earlier)

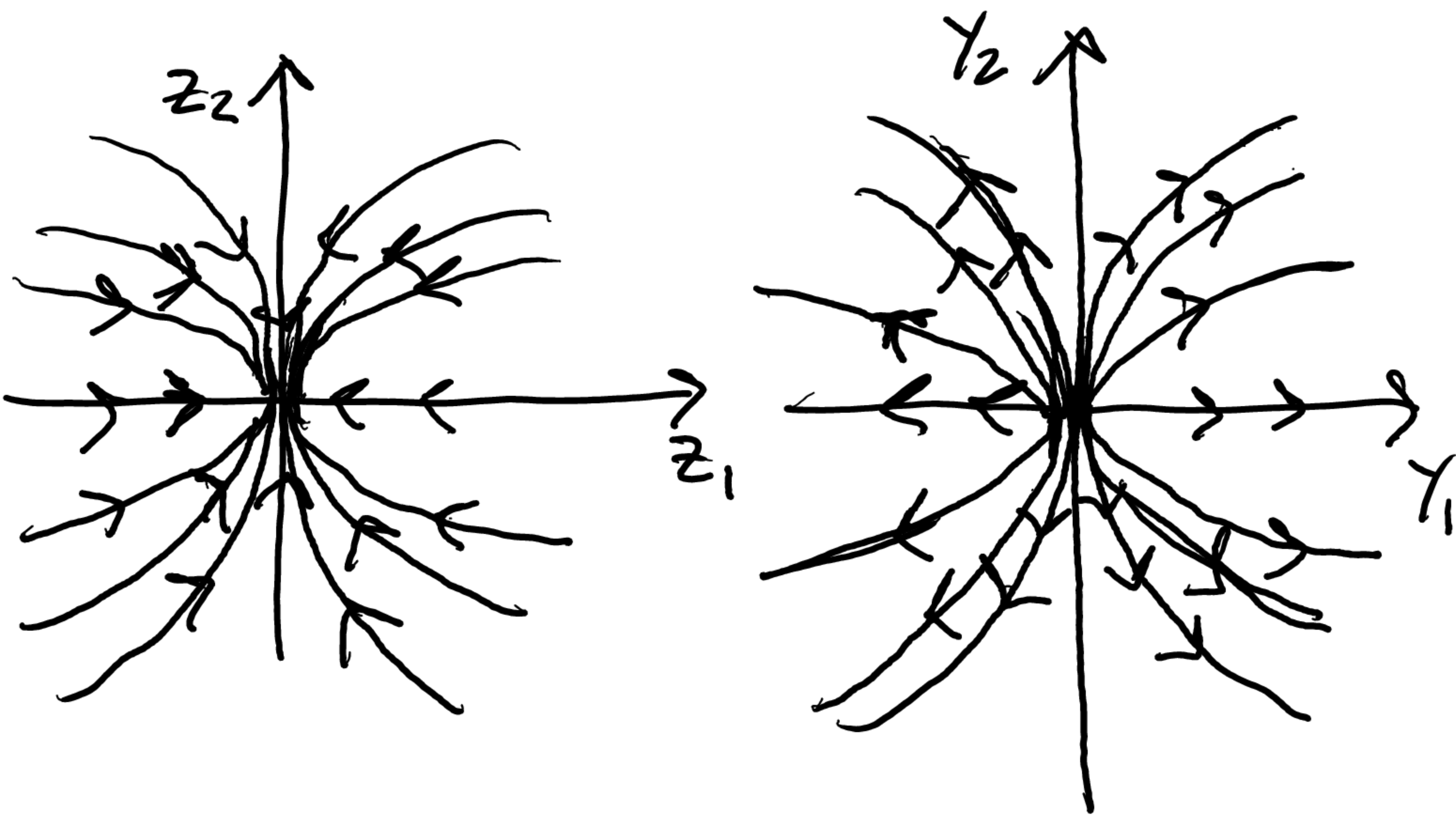


If $\lambda_1 > \lambda_2 > 0$, then 0 is
a source of $I(t) = \begin{bmatrix} c_1 e^{\lambda_1 t} \\ c_2 e^{\lambda_2 t} \end{bmatrix}$.

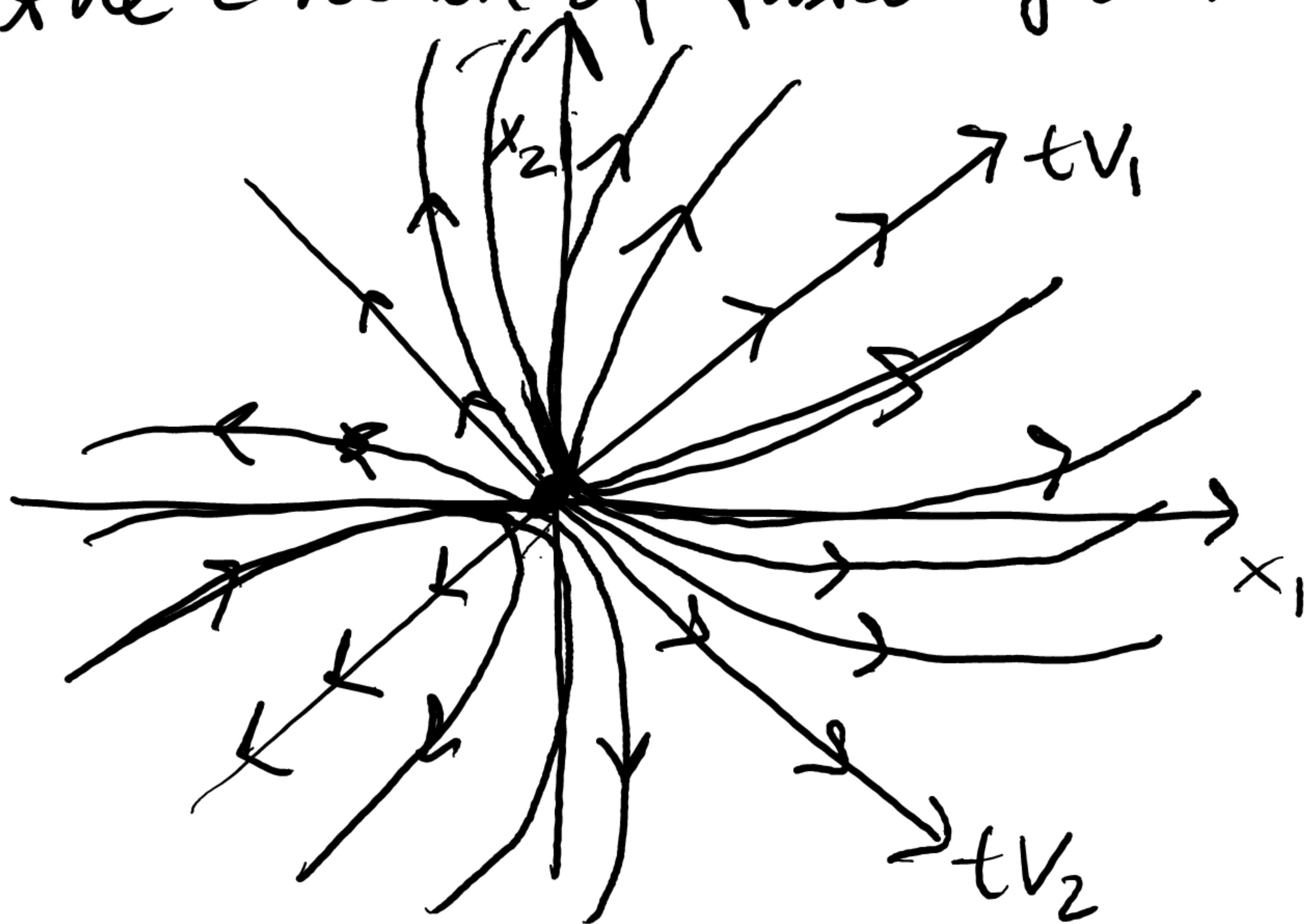
Behavior near 0 :

Note $Z(t) := I(-t)$ has a sink at 0 ,
eigenvals $-\lambda_1 < -\lambda_2 < 0 \Rightarrow Z(t)$ approaches 0
along E_2 -direction (since $-\lambda_2 > -\lambda_1$).

Then $\Sigma(t)$'s direction of fastest growth near 0 is E_1 .



For $\Sigma(t) = c_1 e^{\lambda_1 t} V_1 + c_2 e^{\lambda_2 t} V_2$, V_1 is the direction of fastest growth.



If one eigenvalue equals 0, say $\lambda_1 = 0$,
then the line $\begin{bmatrix} t \\ 0 \end{bmatrix}_{t \in \mathbb{R}}$ are EP for $\mathbb{I}$
and $(tV_1)_{t \in \mathbb{R}}$ are EP for $\mathbb{X}$.

Next week: treat cases $\boxed{2}$ & $\boxed{3}$ and
classification of planar systems.

Exercises

1.1 (Example of ODE that does not have a solution for all times.)

a) Find general solution to $x' = x^2$

b) Discuss the domain of definition of solution

c) Find an ODE that only is defined on $t \in (-1, 1)$.

Sol
a) If $x(0) = 0$, then $x(t) = 0$ is a solution.

If $x(0) \neq 0$, then Sol: $\int \frac{1}{x^2} dx = \int 1 dt$

$$\Rightarrow -\frac{1}{x} = t + C \Rightarrow x(t) = \frac{1}{C - t}$$

b) Valid for $t \in (-\infty, C)$ and

$$\lim_{t \uparrow C} x(t) = \infty.$$

$$c) \text{ Let } X' = \begin{cases} (x+1)^2 & x \geq 0 \\ (x-1)^2 & x < 0 \end{cases}$$

$$X(0) = 0$$

Then Solv for $t \geq 0$,

$$\int \frac{1}{(x+1)^2} dx = \int 1 dt$$

$$-\frac{1}{x+1} = t + C$$

$$\Rightarrow X(t) = \frac{1}{C-t} - 1$$

$$X(0) = 0 \Rightarrow C = 1 \text{ \& } X(t) = \frac{t}{1-t}$$

Solv for $t < 0$

$$\int \frac{1}{(x-1)^2} dx = \int 1 dt$$

$$-\frac{1}{x-1} = t + C \Rightarrow X(t) = \frac{1}{C-t} + 1$$

$$X(0) = 0 \Rightarrow C = -1 \text{ \& } X(t) = \frac{t}{1+t}$$

Summary:

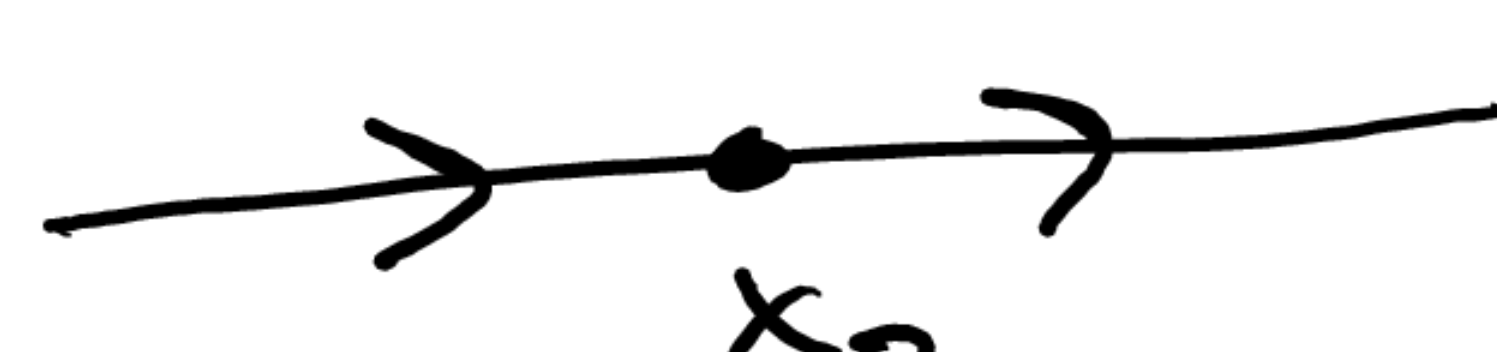
$$x(t) = \frac{t}{1-|t|} \quad t \in (-1, 1)$$

$$\left(\lim_{|t| \uparrow 1} |x(t)| = \infty \right).$$

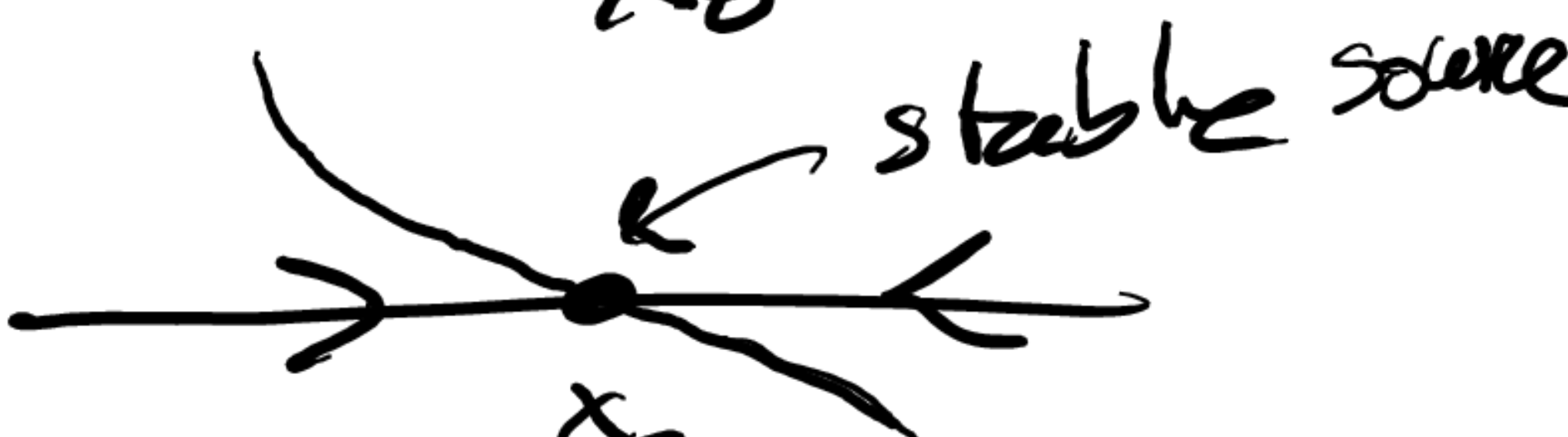
1.13 Let $x' = f(x)$ with EP at x_0

a) Suppose $f'(x_0) = 0$. What can you say about the behavior of solutions near x_0 ?

Sol x_0 can be: stable or unstable:

$$f(x) = (x - x_0)^2 \Rightarrow$$


x_0

$$f(x) = -(x - x_0)^3 \Rightarrow$$


x_0

$$f(x) = (x - x_0)^3 \Rightarrow$$


x_0

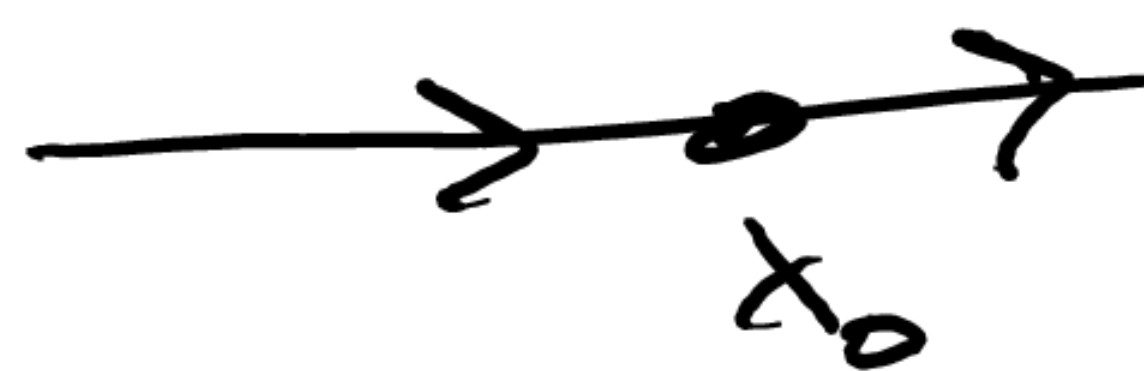
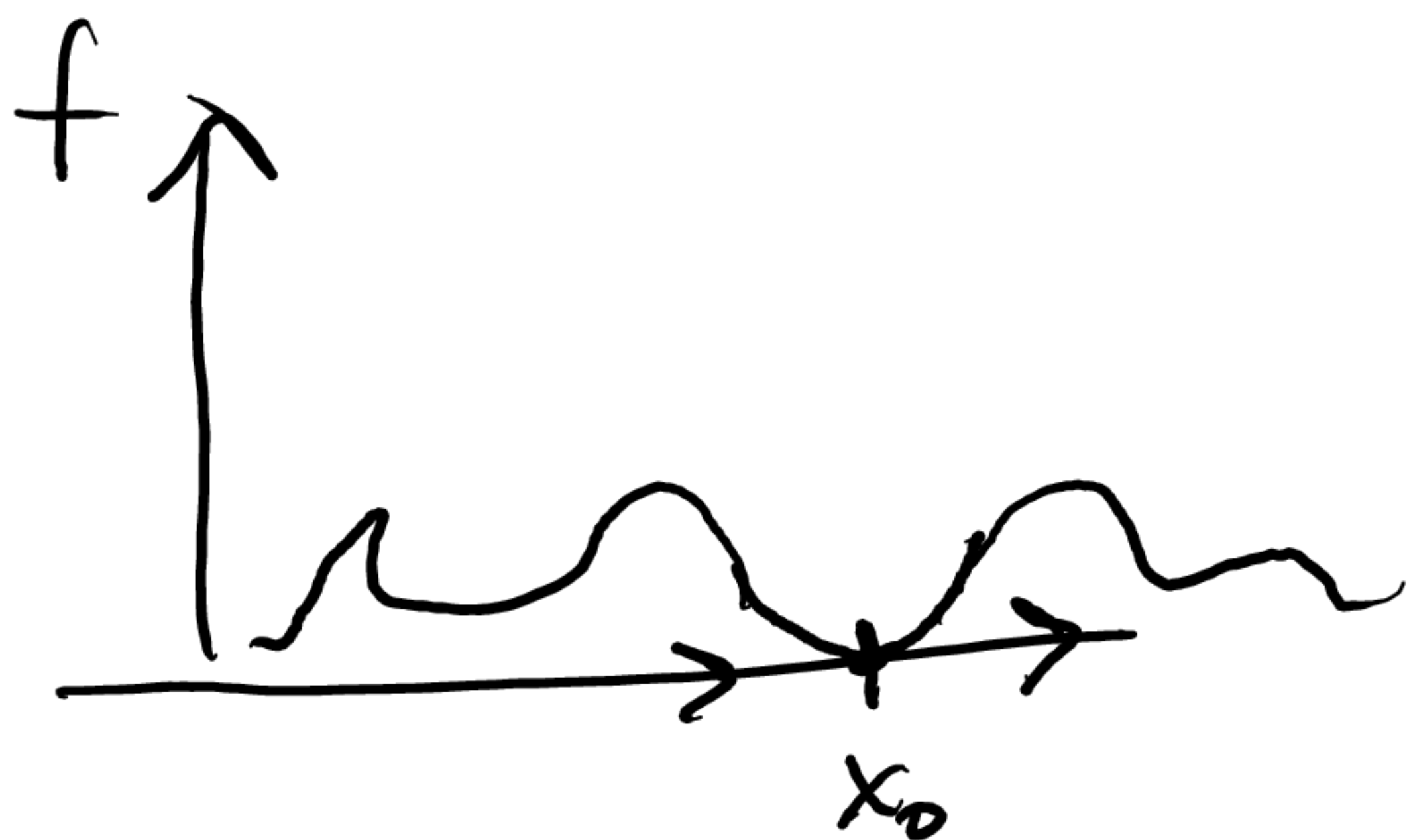
b) Suppose $f'(x_0) = 0$ & $f''(x_0) \neq 0$
what can you say now?

Sol By Mean-Val formula

$$f(x) = f(x_0) + f'(x_0)(x - x_0) + \frac{1}{2}f''(x_0)(x - x_0)^2 + o((x - x_0)^2)$$

$$= \underbrace{f''(x_0) \frac{(x - x_0)^2}{2}}_{\text{dominates for } x \text{ near } x_0} + O((x - x_0)^3)$$

If $f''(x_0) > 0$, then



so x_0 unstable and $x' \geq 0$ near x_0

If $f''(x_0) < 0$, then ~~not~~

so x_0 unstable and $x' \leq 0$ near x_0 .

2.3

(a) $F(\underline{x}) = \begin{bmatrix} x_1 + 2x_2 \\ 3x_2 \end{bmatrix}$ points in E_1 direction

along line $\begin{bmatrix} t \\ 0 \end{bmatrix}$. Only matches 4.

(b) $F(\underline{x}) = \begin{bmatrix} x_1 + 2x_2 \\ 3x_1 + 6x_2 \end{bmatrix}$

$\Rightarrow F(\underline{x}) \perp \begin{bmatrix} 1 \\ 3 \end{bmatrix}$ except when $x_1 = -2x_2$.
only matches 2.

(c) $F(\underline{x}) = \begin{bmatrix} x_1 + 2x_2 \\ x_1 \end{bmatrix}$ points in E_1 direction

along line $\begin{bmatrix} 0 \\ t \end{bmatrix}$. only matches 1.

(d) $F(\underline{x}) = \begin{bmatrix} x_1 + 2x_2 \\ 3(x_1 - x_2) \end{bmatrix}$ Should point in E_1 -direction
along line $\begin{bmatrix} t \\ t \end{bmatrix}$.

Only matches 3.

2.14. Prove that if $A \in \mathbb{R}^{2 \times 2}$ has real distinct eigenvalues λ_1, λ_2 , then its eigenvectors $v_1, v_2 \in \mathbb{R}^2$ are linearly independent.

Sol Suppose not. Then $v_2 = c v_1$ so

$$\begin{aligned} \text{that } A v_2 &= A c v_1 = c A v_1 \\ &= c \lambda_1 v_1 = \lambda_1 c v_1 \\ &= \lambda_1 v_2 \end{aligned}$$

$\Rightarrow \lambda_1 = \lambda_2$ contradicting that λ_1 and λ_2 are distinct. $\square$