

Recall that for

$$\mathbf{X}' = A\mathbf{X} \quad \text{with } A \in \mathbb{R}^{2 \times 2}, \quad \text{and}$$

any similar matrix $B = P^{-1}AP$

$$\mathbf{Y}(t) \text{ solves } \mathbf{Y}' = B\mathbf{Y}$$

$$\Leftrightarrow \mathbf{X}(t) = P\mathbf{Y} \text{ solves } \mathbf{X}' = A\mathbf{X}.$$

(Find solution $\mathbf{X}$ by first solving
 $\mathbf{Y}' = B\mathbf{Y}$ for suitable P .)

$A \in \mathbb{R}^{2 \times 2}$ is one of three types:

1 A has two real eigenvalues
 λ_1, λ_2 and lin. independent
eigen vectors $\mathbf{v}_1, \mathbf{v}_2$

Then EP 0 is a sink if $\lambda_1, \lambda_2 < 0$,
source if $\lambda_1, \lambda_2 > 0$ and saddle if $\lambda_1, \lambda_2 < 0$.

[2] A has one repeated real eigenvalue λ , and eigenspace $\text{Ker}(A - \lambda I)$ has dimension 1.

[3] A has a pair of complex eigenvalues $\lambda_1 = a + ib$, $\lambda_2 = a - ib$.

Proposition 1 In case [2] A is similar to $\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}$.

Proof Let $v \in \mathbb{R}^2$ be eigenvector s.t. $Av = \lambda v$. Find a generalized eigenvector i.e. a w solving $(A - \lambda I)w = v$.

Then $w \notin \text{Ker}(A - \lambda I) = \{c v \mid c \in \mathbb{R}\}$ so v & w are linearly independent.

Observe $Aw = v + \lambda w$

(proof that such a W exists:

As $\ker(A - \lambda I) = \text{span}(V) \subset \mathbb{R}^2$ has $\dim 1$, there exists a $U \in \mathbb{R}^2$ s.t.

$\ker(A - \lambda I)^\perp = \text{span}(U)$. Note $U^T V = 0$

& $\mathbb{R}^2 = \text{span}(V) \oplus \text{span}(U)$.

Consider $(A - \lambda I)U = \alpha V + \beta U$

for some $\alpha, \beta \in \mathbb{R}$. $\alpha \neq 0$, otherwise U is an eigenvector. And $\beta \neq 0$, otherwise

$$Z = \frac{\alpha}{\beta} V + U$$

$$\Rightarrow (A - \lambda I)Z = (A - \lambda I)U = \alpha V + \beta U = \beta Z$$

$$\Rightarrow AZ = (\lambda + \beta)Z \quad \text{so } Z \text{ is then an eigenvector.}$$

Hence $W = \frac{U}{\alpha}$ yields $(A - \lambda I)W = V$.

set $P = [V \ W]$. Note that

$$I = P^{-1}P = [P^{-1}V \ P^{-1}W] = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix},$$

and $AP = [AV \ AW] = [\lambda V \ (V + 2W)]$

Hence

$$P^{-1}AP = [P^{-1}(\lambda V) \ P^{-1}(V + 2W)]$$

$$= [\lambda E_1 \ (E_1 + 2E_2)] = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}.$$



Ex 1 $A = \begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}$. Has eigenvalue 4

and $\text{Ker}(A - 4I) = \text{Ker} \begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} = \text{span} \left(\begin{bmatrix} 1 \\ -1 \end{bmatrix} \right)$
has dimension 1.

Generalized eigenvalue

$$(A - 4I)W = \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

$$\begin{bmatrix} 1 & 1 \\ -1 & -1 \end{bmatrix} \begin{bmatrix} w_1 \\ w_2 \end{bmatrix} = \begin{bmatrix} 1 \\ -1 \end{bmatrix} \text{ has solution } W = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

$$P = [V \ W] = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix} \text{ and } P^{-1}AP = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}.$$

To solve $\mathbf{X}' = A\mathbf{X}$ in case [2] we find P as above, first solve

$$\mathbf{Y}' = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \mathbf{Y} \quad (1) \quad \text{and set}$$

$$\mathbf{X}(t) = P\mathbf{Y} = Y_1(t)V + Y_2(t)W$$

Note E, P, O will be of same type for $\mathbf{X}$ as for $\mathbf{Y}$.

$$\text{Solution of (1): } \begin{aligned} Y_1' &= \lambda Y_1 + Y_2 \\ Y_2' &= \lambda Y_2 \end{aligned}$$

yields 1. $Y_2(t) = C_2 e^{\lambda t}$

2. Insert in first eq:

$$Y_1'(t) = \lambda Y_1(t) + C_2 e^{\lambda t} \quad \left| \begin{array}{l} \text{integrating} \\ \text{factor} \\ e^{-\lambda t} \end{array} \right.$$

$$\frac{d}{dt} (Y_1 e^{-\lambda t}) = C_2$$

$$\Rightarrow Y_1 e^{-\lambda t} = C_2 t + C_1$$

$$Y_1(t) = C_2 t e^{\lambda t} + C_1 e^{\lambda t}$$

General solution

$$\mathbf{Y}(t) = C_1 e^{\lambda t} \begin{bmatrix} 1 \\ 0 \end{bmatrix} + C_2 e^{\lambda t} \begin{bmatrix} t \\ 1 \end{bmatrix}$$

If $\lambda < 0$, then $\lim_{t \rightarrow \infty} |\mathbf{Y}(t)| = 0$

and 0 is a sink

If $\lambda > 0$, then $\lim_{t \rightarrow \infty} |\mathbf{I}(t)| = \infty$

and O is a source.

If $\lambda = 0$, then $\mathbf{I}(t) = \begin{bmatrix} c_1 + c_2 t \\ c_2 \end{bmatrix}$ and

O is unstable but neither source nor sink.

Ex 2 Find general solution to

$$\mathbf{X}' = A\mathbf{X} \text{ with } A = \begin{bmatrix} 5 & 1 \\ -1 & 3 \end{bmatrix}.$$

Sol: A is similar to $B = P^{-1}AP = \begin{bmatrix} 4 & 1 \\ 0 & 4 \end{bmatrix}$

using $P = \begin{bmatrix} 1 & 0 \\ -1 & 1 \end{bmatrix}$ by Ex 1.

$\mathbf{I}' = B\mathbf{I}$ has general solution

$$\mathbf{I}(t) = \begin{bmatrix} (c_1 + c_2 t)e^{4t} \\ c_2 e^{4t} \end{bmatrix}$$

$$\Rightarrow \mathbf{X}(t) = P\mathbf{I}(t) = \begin{bmatrix} (c_1 + c_2 t)e^{4t} \\ (c_2(1-t) - c_1)e^{4t} \end{bmatrix}$$

Case [3] A has eigenvalues

$$\lambda = \alpha + i\beta, \quad \bar{\lambda} = \alpha - i\beta \quad \text{with } \beta \neq 0.$$

Then there exists a $V \in \mathbb{C}^2$ s.t.

$$AV = \lambda V \quad \text{and} \quad A\bar{V} = \bar{\lambda} V.$$

$$\text{Set } V_1 = \operatorname{Re}(V) \in \mathbb{R}^2, \quad V_2 = \operatorname{Im}(V) \in \mathbb{R}^2,$$

$$\text{to obtain } V = V_1 + iV_2.$$

Claim 1 $V_2 \neq 0$ Proof: If not, $AV = \lambda V$ would become $\underbrace{AV_1}_{\text{real}} = \underbrace{(\alpha + i\beta)V_1}_{\text{complex}}$: a contradiction.

Claim 2 V_1 & V_2 are linearly independent.

Proof Suppose $V_1 = cV_2$ for some $c \in \mathbb{R}$.

Then $A(V_1 + iV_2) = (c + i)AV_2$, and also

$$\begin{aligned} A(V_1 + iV_2) &= \lambda(V_1 + iV_2) \\ &= (\alpha + i\beta)(c + i)V_2 \end{aligned}$$

Leads to contradiction $\underbrace{AV_2}_{\text{real}} = \underbrace{(\alpha + i\beta)V_2}_{\text{complex}}$

(Claims 1 & 2 also imply that $V_1 \neq 0$.)

Equations for V_1 & V_2 : Expanding

$$Av = \lambda v \text{ yields}$$

$$A(V_1 + iV_2) = (\alpha + i\beta)(V_1 + iV_2)$$

$\Rightarrow$

$$AV_1 + iAV_2 = \alpha V_1 - \beta V_2 + i(\beta V_1 + \alpha V_2)$$

$$\text{Real part: } AV_1 = \alpha V_1 - \beta V_2 = [V_1 \ V_2] \begin{bmatrix} \alpha \\ -\beta \end{bmatrix}$$

$$\text{Imaginary part: } AV_2 = \beta V_1 + \alpha V_2 = [V_1 \ V_2] \begin{bmatrix} \beta \\ \alpha \end{bmatrix}$$

Similarity transformation:

$$\text{For } P = [V_1 \ V_2], \quad AP = \underbrace{[V_1 \ V_2]}_P \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

Hence,

$$P^{-1}AP = P^{-1}P \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} =: B$$

B is called the Jordan form/canonical form of A (in all three cases).

Let $\underline{Y}(t)$ solves $\underline{Y}' = B \underline{Y} = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \underline{Y}$.

Then $\underline{X} = P \underline{Y} = Y_1(t) V_1 + Y_2(t) V_2$

solves $\underline{X}' = A \underline{X}$,

and stability of $\underline{Y}^* = 0$ for $\underline{Y}(t)$ coincides with stability of $\underline{X}^* = 0$ for $\underline{X}(t)$.

Remains to study

$$\underline{Y}' = \underbrace{\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}}_B \underline{Y} \quad \alpha \in \mathbb{R}, \beta \neq 0 \quad (1)$$

Eigenvalues: $\lambda = \alpha + i\beta$ and $\bar{\lambda}$.

$$\text{Eigenvectors: } (B - \lambda I) V = \beta \begin{bmatrix} -i & 1 \\ -1 & -i \end{bmatrix} V = 0$$

$$V = \begin{bmatrix} 1 \\ i \end{bmatrix} \quad \text{and} \quad \bar{V}.$$

Claim 3 $I(t) = e^{\lambda t} V$ and $\bar{I} = e^{\bar{\lambda} t} \bar{V}$
are complex solutions of (1),
where $\frac{d}{dt} I(t) = \lambda e^{\lambda t} V$

Proof $I'(t) = e^{\lambda t} \lambda V = e^{\lambda t} B V$
 $= B(e^{\lambda t} V) = B I.$

And $\overline{I}'(t) = \overline{B I} = B \bar{I}.$ $\square$

Claim 4 For any $c_1, c_2 \in \mathbb{C}$,
 $Z(t) = c_1 I(t) + c_2 \bar{I}(t)$ solves (1)
(possibly a complex solution).

(same proof as for case 1).

Note $I = e^{\alpha t}$ $V = e^{\alpha t} \cdot e^{i\beta t} \begin{bmatrix} 1 \\ i \end{bmatrix}$

$$= e^{\alpha t} (\cos \beta t + i \sin \beta t) \begin{bmatrix} 1 \\ i \end{bmatrix}$$

$$= e^{\alpha t} \underbrace{\begin{bmatrix} \cos \beta t \\ -\sin \beta t \end{bmatrix}}_{U(t)} + i e^{\alpha t} \underbrace{\begin{bmatrix} \sin \beta t \\ \cos \beta t \end{bmatrix}}_{W(t)}$$

$$= U(t) + W(t)$$

Exercise Show that $\lambda e^{\lambda t} V = \underbrace{U'(t) + i W'(t)}_{\substack{\text{derivatives} \\ \text{of real-valued} \\ \text{fns}}}$

By claim 4, we have real solutions

$$U(t) = \frac{\underline{I}(t) + \overline{\underline{I}}(t)}{2},$$

$$W(t) = \frac{\underline{I}(t) - \overline{\underline{I}}(t)}{2i}.$$

Theorem The general real solution
of $Y' = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} Y$ is

$$Y(t) = \mu U(t) + \gamma W(t) = \mu e^{\alpha t} \begin{bmatrix} \cos \beta t \\ -\sin \beta t \end{bmatrix} + \gamma e^{\alpha t} \begin{bmatrix} \sin \beta t \\ \cos \beta t \end{bmatrix}$$

for $\mu, \gamma \in \mathbb{R}$.

Proof Remains to show generality.

Suppose $Z(t) = \begin{bmatrix} f(t) \\ g(t) \end{bmatrix}$ is a real solution.

$$\text{Then } \begin{bmatrix} f'(t) \\ g'(t) \end{bmatrix} = Z'(t) = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \begin{bmatrix} f \\ g \end{bmatrix} = \begin{bmatrix} \alpha f + \beta g \\ -\beta f + \alpha g \end{bmatrix}$$

$$\text{and } h(t) := (f(t) + ig(t))e^{-\bar{\lambda}t}$$

$$\Rightarrow h'(t) = [-\bar{\lambda}(f + ig) + f' + ig']e^{-\bar{\lambda}t} = 0$$

$$\Rightarrow f(t) + ig(t) = (\mu + i\gamma)e^{\bar{\lambda}t} \quad \text{for some } \mu, \gamma \in \mathbb{R}$$

$$\Rightarrow \begin{bmatrix} f \\ g \end{bmatrix} = \mu e^{\alpha t} \begin{bmatrix} \cos \beta t \\ -\sin \beta t \end{bmatrix} + \gamma e^{\alpha t} \begin{bmatrix} \sin \beta t \\ \cos \beta t \end{bmatrix} \quad \blacksquare$$

Given $\gamma(t_0) = \gamma_0 \in \mathbb{R}^2$, the solution is unique (as $\text{span}(U(t_0), W(t_0)) = \mathbb{R}^2$).

Next lecture: stability of EP for case 3, and dynamical classification of planar linear systems.