

Lecture 8 11/2

Planar linear systems

$$\mathbf{X}' = A\mathbf{X} \quad \text{for } A \in \mathbb{R}^{2 \times 2}.$$

Case [B] When A has complex eigenpairs (λ, V) and $(\bar{\lambda}, \bar{V})$ with

$$\lambda = \alpha + i\beta \quad \text{for some } \beta \neq 0,$$

$$\text{then } B = P^{-1}AP = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

$$\text{for } P = [V_1 \ V_2] \quad \text{where } V = V_1 + iV_2.$$

$\mathbf{Y}' = B\mathbf{Y}$ has general (real) solution

$$\mathbf{Y}(t) = \mu e^{\alpha t} \begin{bmatrix} \cos \beta t \\ -\sin \beta t \end{bmatrix} + \gamma e^{\alpha t} \begin{bmatrix} \sin \beta t \\ \cos \beta t \end{bmatrix}$$

$$\mu, \gamma \in \mathbb{R}.$$

General solution for $\bar{X}$ becomes

$$\bar{X} = P \bar{Y}(t; \mu, \gamma) = \gamma_1(t) V_1 + \gamma_2(t) V_2 \quad \mu, \gamma \in \mathbb{R}.$$

(If $\bar{X}' = A \bar{X}$ has a solution $\bar{X}$, then

$$\bar{Y} = P^{-1} \bar{X} \text{ solves } \bar{Y}' = B \bar{Y} \text{ (Exercise),}$$

hence $\bar{Y} = \bar{Y}(t; \mu, \gamma)$ for some μ, γ .)

Stability of EP $\bar{X}^* = 0$

Stability of $\bar{X}^* = 0$ for $\bar{X}(t) = P \bar{Y}(t)$
coincides with stability of $\bar{Y}^* = 0$
for $\bar{Y}(t)$.

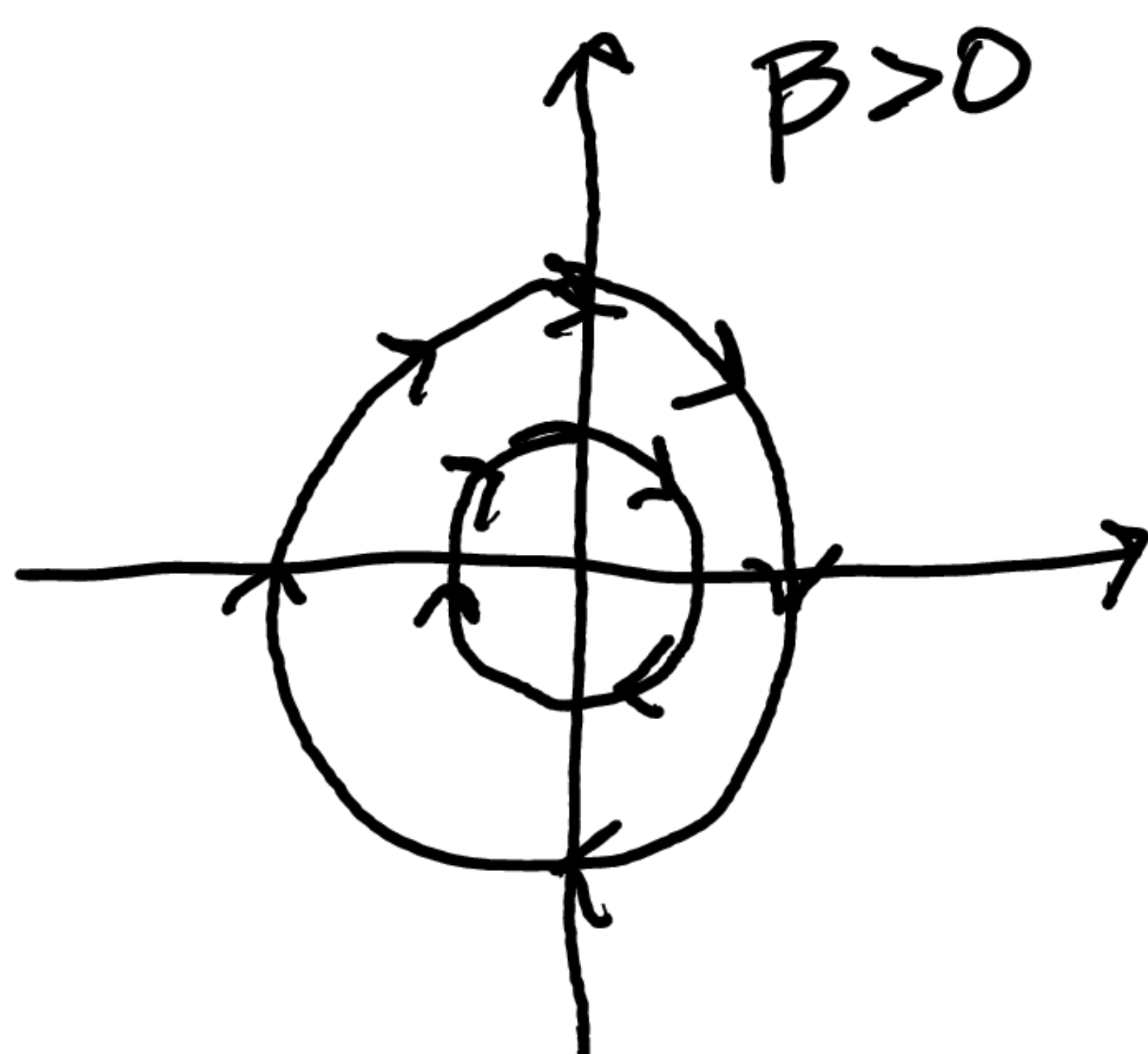
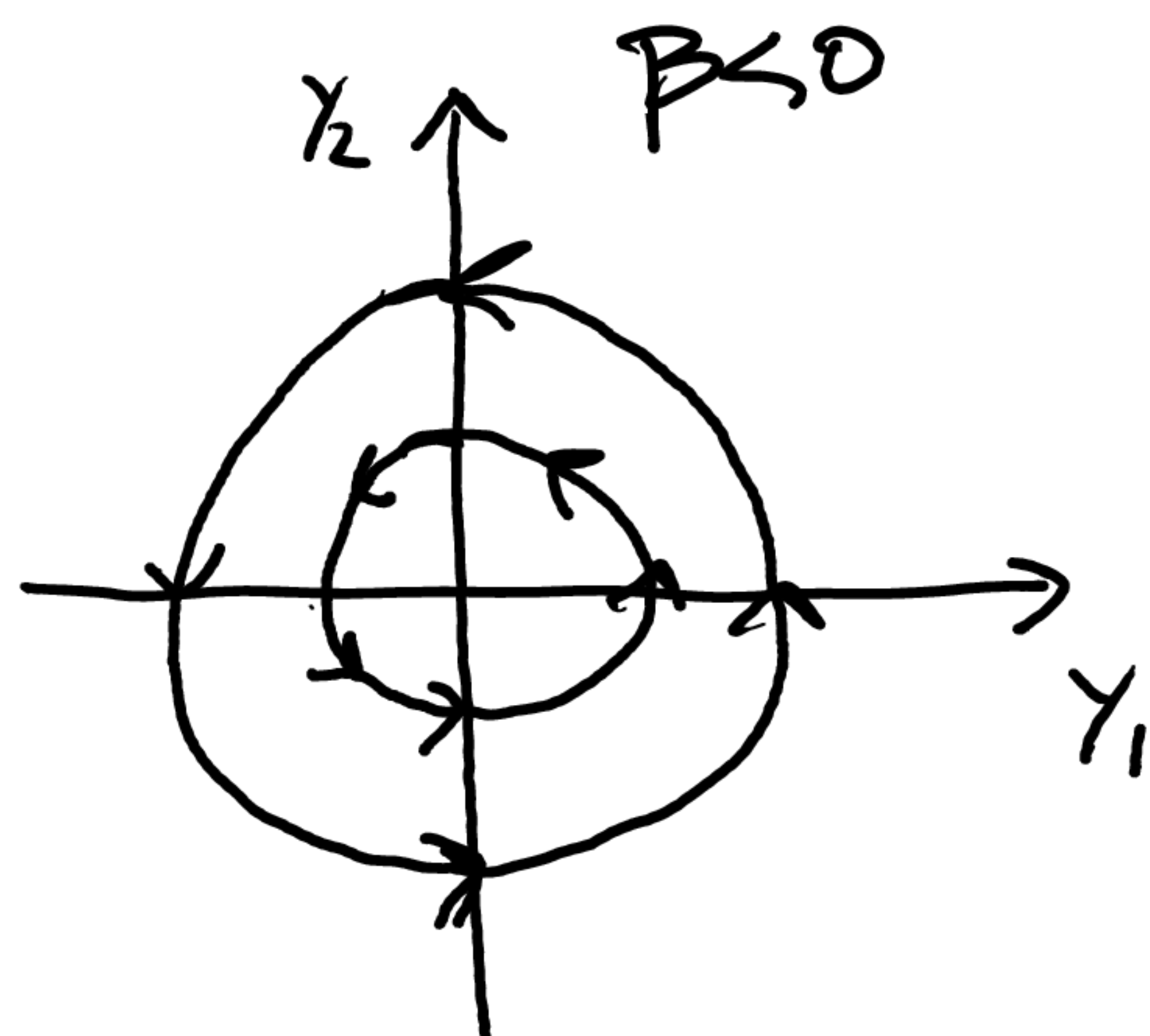
If $\alpha = 0$ then

$$\bar{Y}(t) = \mu \begin{bmatrix} \cos \beta t \\ -\sin \beta t \end{bmatrix} + \gamma \begin{bmatrix} \sin \beta t \\ \cos \beta t \end{bmatrix}$$

rotates around the center $\bar{Y}^* = 0$

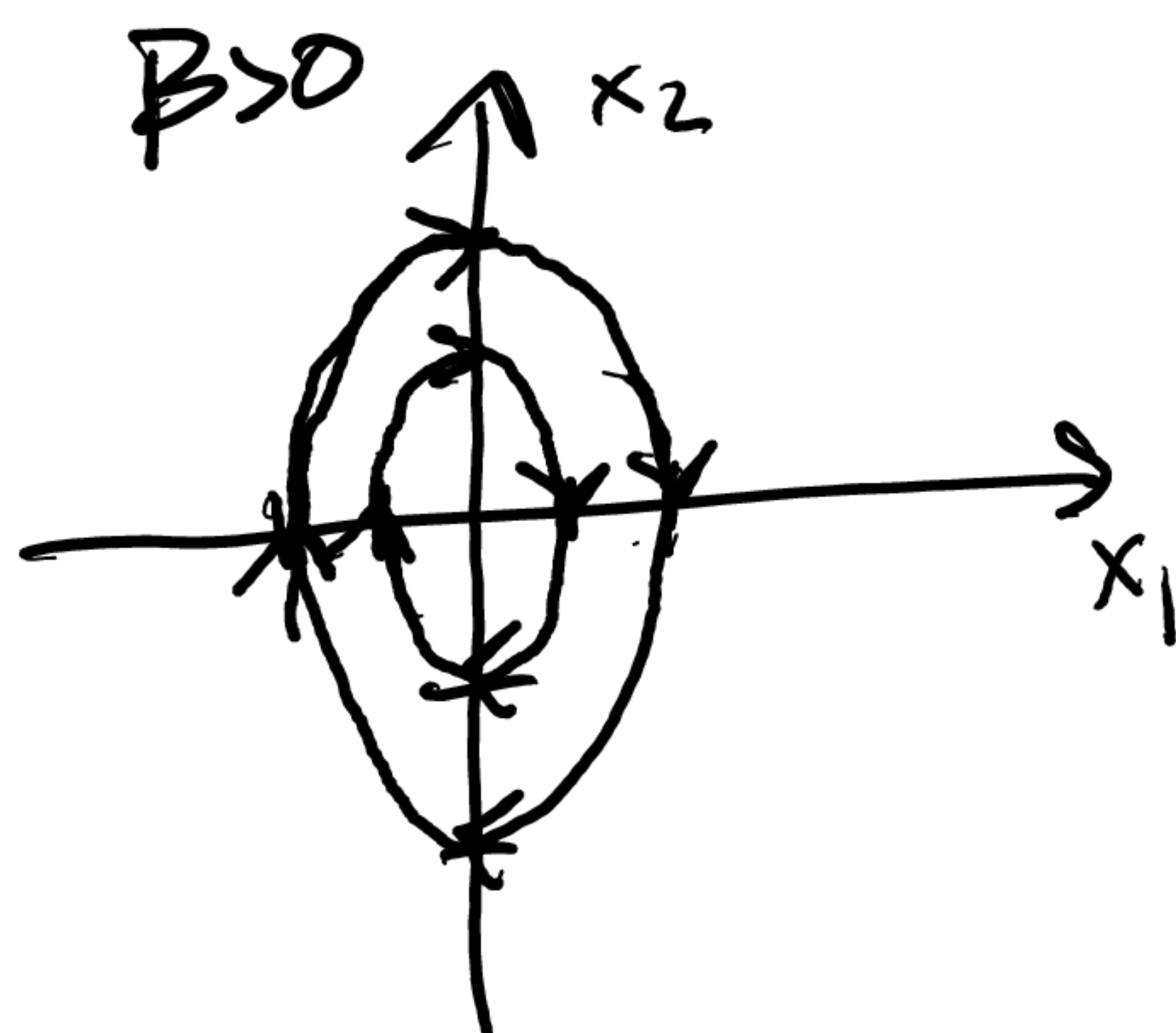
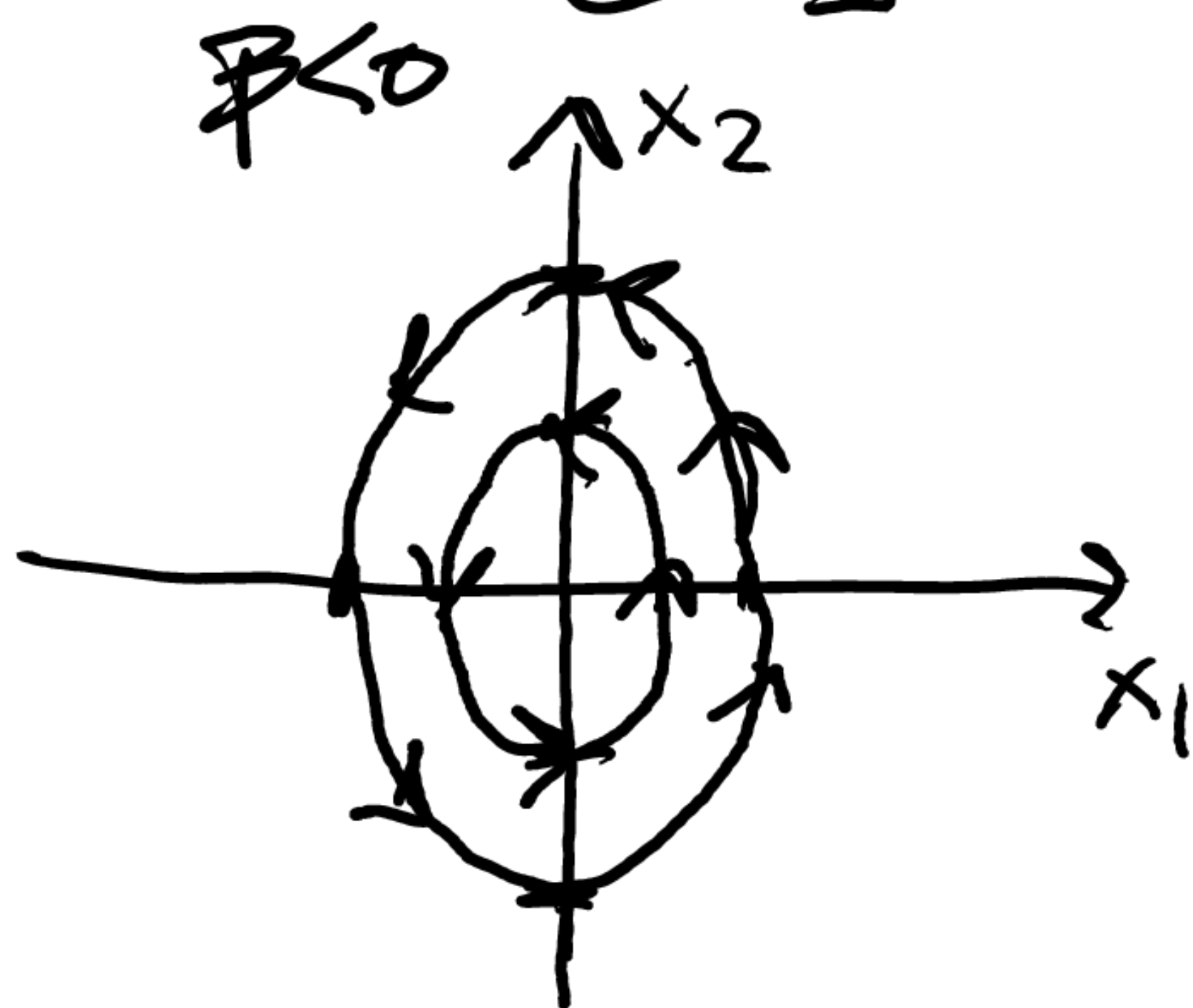
Phase portrait

$$\dot{Y} = \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix} Y = \beta \begin{bmatrix} y_2 \\ -y_1 \end{bmatrix}$$



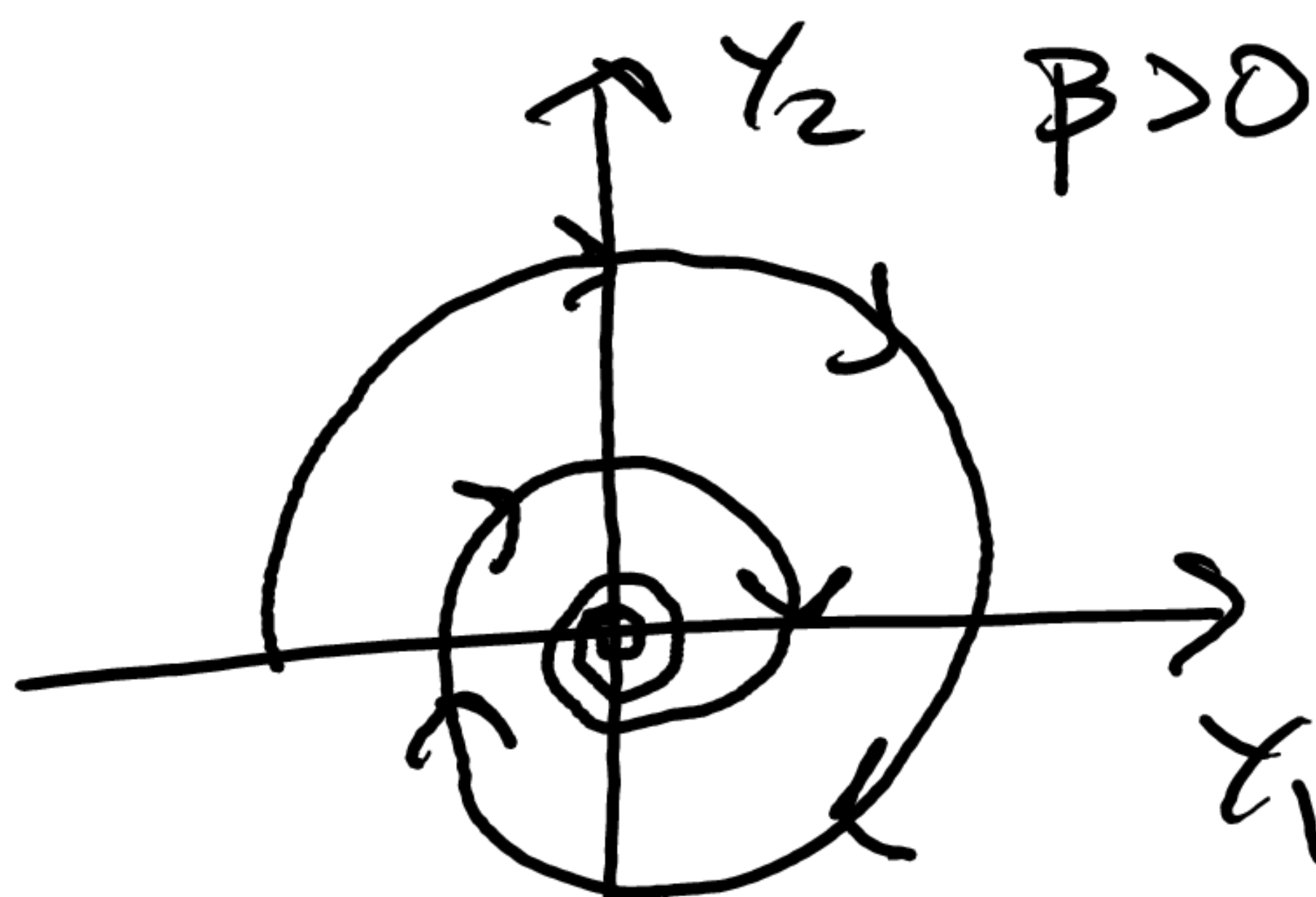
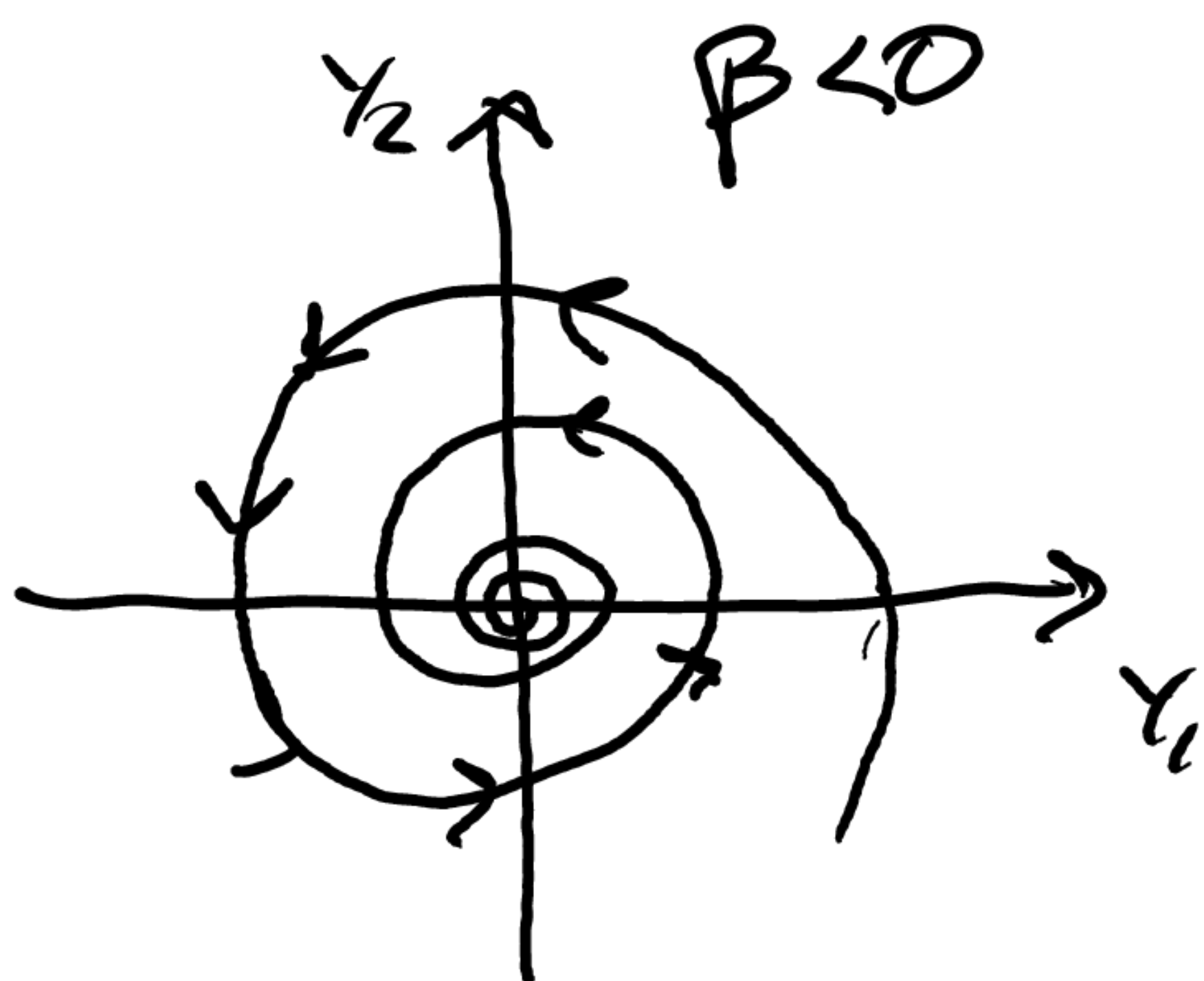
Coordinate transformation $Z = PY$ may scale, rotate and reflect.

Ex $P = \begin{bmatrix} 1 & 0 \\ 0 & 2 \end{bmatrix} \Rightarrow Z = \begin{bmatrix} y_1 \\ 2y_2 \end{bmatrix}$ and



For $\alpha = 0$, the EP Z^* is called a center. It is stable, but not asymptotically stable.

If $\alpha < 0$, then $\lim_{t \rightarrow \infty} \mathbf{Y}(t) = \mathbf{0}$



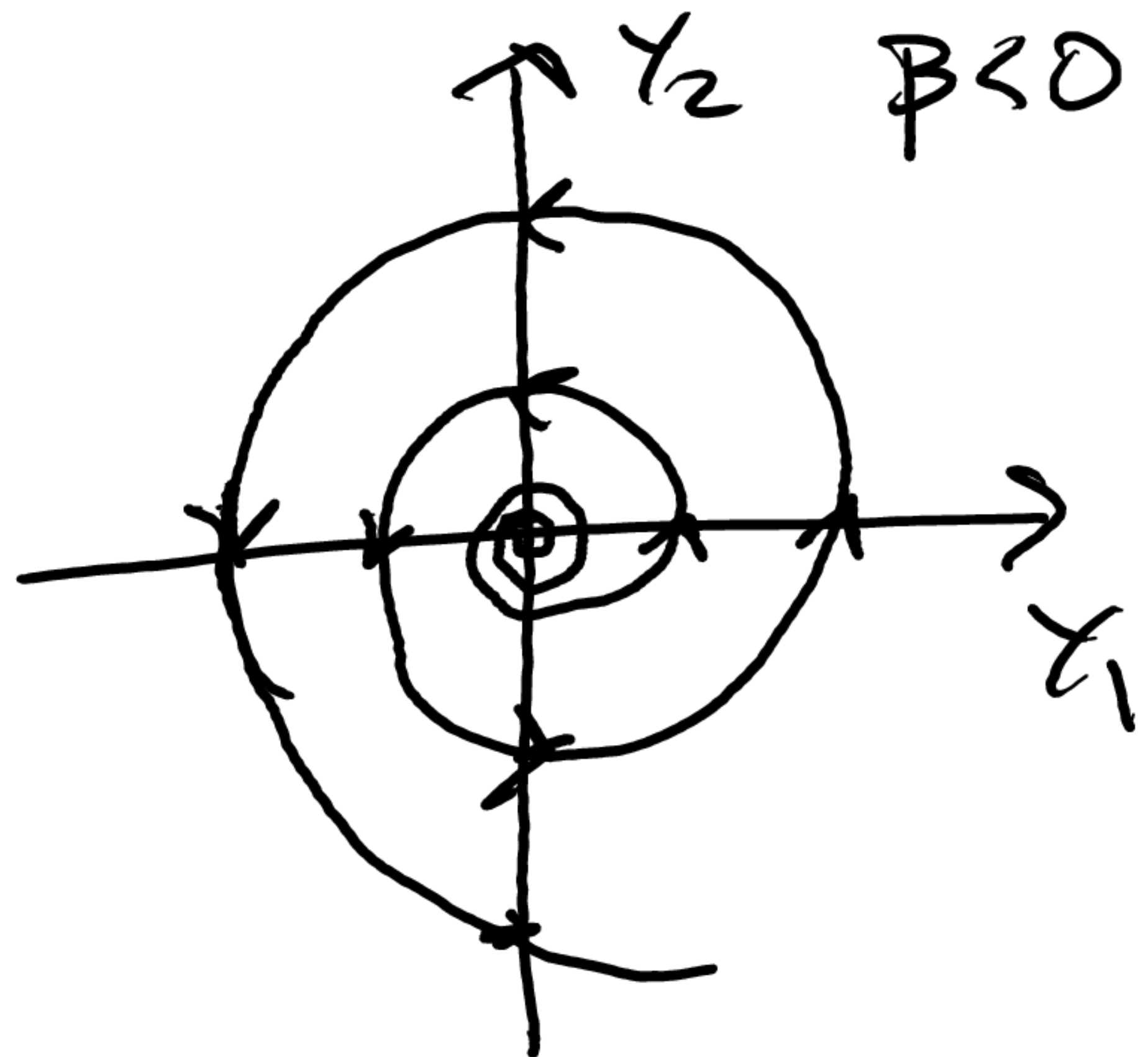
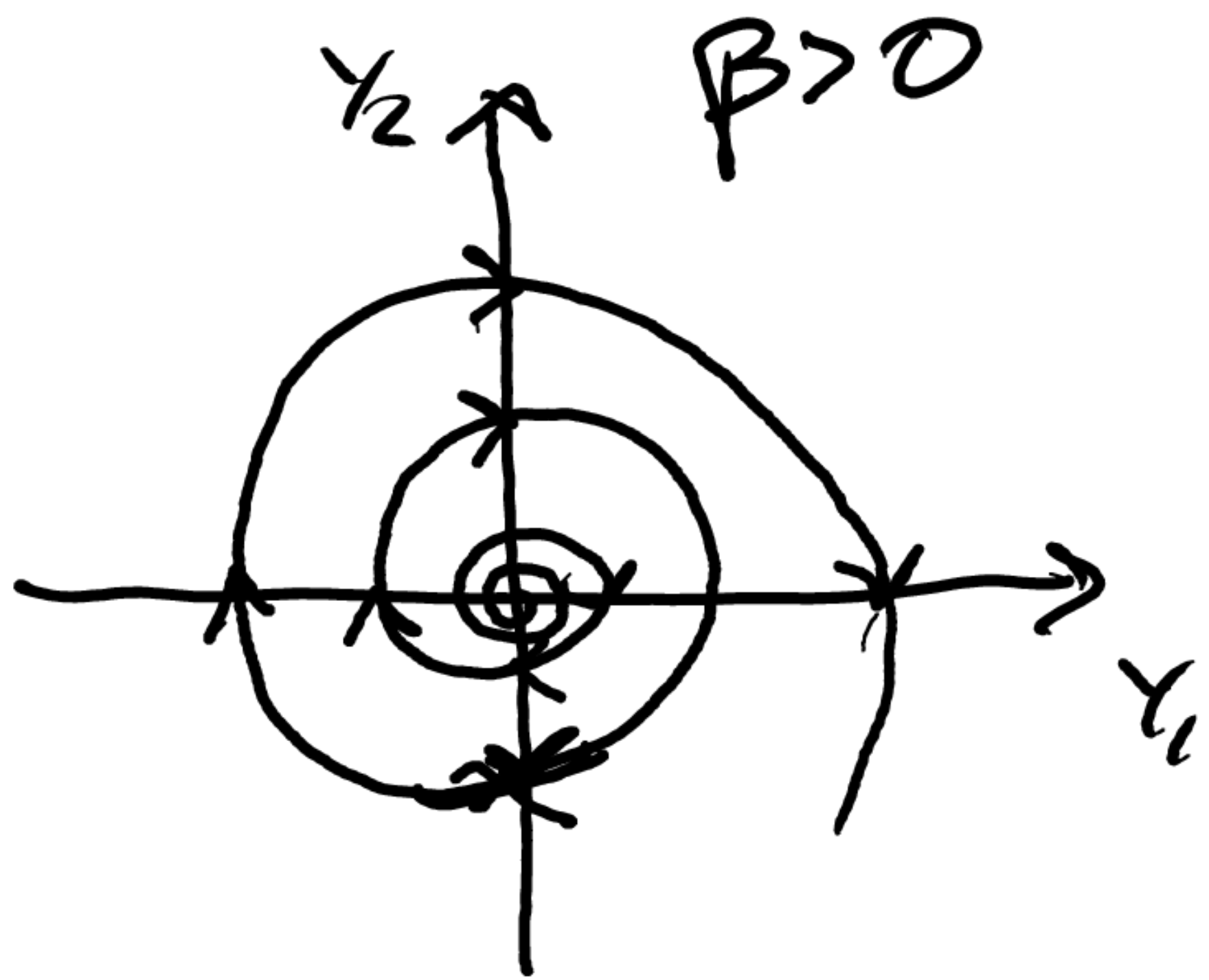
(See direction of rotation from)

$$\mathbf{Y}' = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} \alpha y_1 + \beta y_2 \\ -\beta y_1 + \alpha y_2 \end{bmatrix}$$

EP $\mathbf{0}$ is then called a spiral sink.

It is A-stable

If $\alpha > 0$ and $\gamma(0) \neq 0$, then $\lim_{t \rightarrow \infty} |\gamma(t)| = \infty$



EP 0 is unstable.

Summary $\gamma' = \begin{bmatrix} \alpha & \beta \\ \beta & \alpha \end{bmatrix} \gamma$

α	EP 0 name	Phase portrait
0	center	
< 0	spiral sink	
> 0	spiral source	

Trace determinant plane

$$\mathbf{X}' = \underbrace{\begin{bmatrix} a & b \\ c & d \end{bmatrix}}_A \mathbf{X}$$

$$\begin{aligned} \text{char eq: } \det(A - \lambda I) &= \lambda^2 - \operatorname{tr} A \lambda + \det A = 0 \\ &= (\lambda - \lambda_+) (\lambda - \lambda_-) \end{aligned}$$

$$\text{where } \lambda_{+/-} = \frac{T \pm \sqrt{T^2 - 4D}}{2} \quad \text{with}$$

$$T := \operatorname{tr} A \quad \text{and} \quad D := \det A.$$

Relation to dynamics

Consider $D > 0$: If $T^2 < 4D$, then

$$\lambda_{+/-} = \frac{T}{2} \pm \frac{\sqrt{T^2 - 4D}}{2} \quad \text{are complex}$$

EP 0 is spiral sink if $T < 0$,
center if $T = 0$, and spiral source if $T > 0$.

If $T^2 = 4D$, then $\lambda_+ = \lambda_-$ (repeated ev)

If $T^2 > 4D$, then $\lambda_+, \lambda_- \in \mathbb{R}$ are distinct.

$$\lambda_{\pm} = \frac{T}{2} + \frac{1}{2} \underbrace{\sqrt{T^2 - 4D}}_{< T}$$

so $\lambda_{+/-}$ have same sign as T .

EPD is sink if $T < 0$ and
source if $T > 0$.

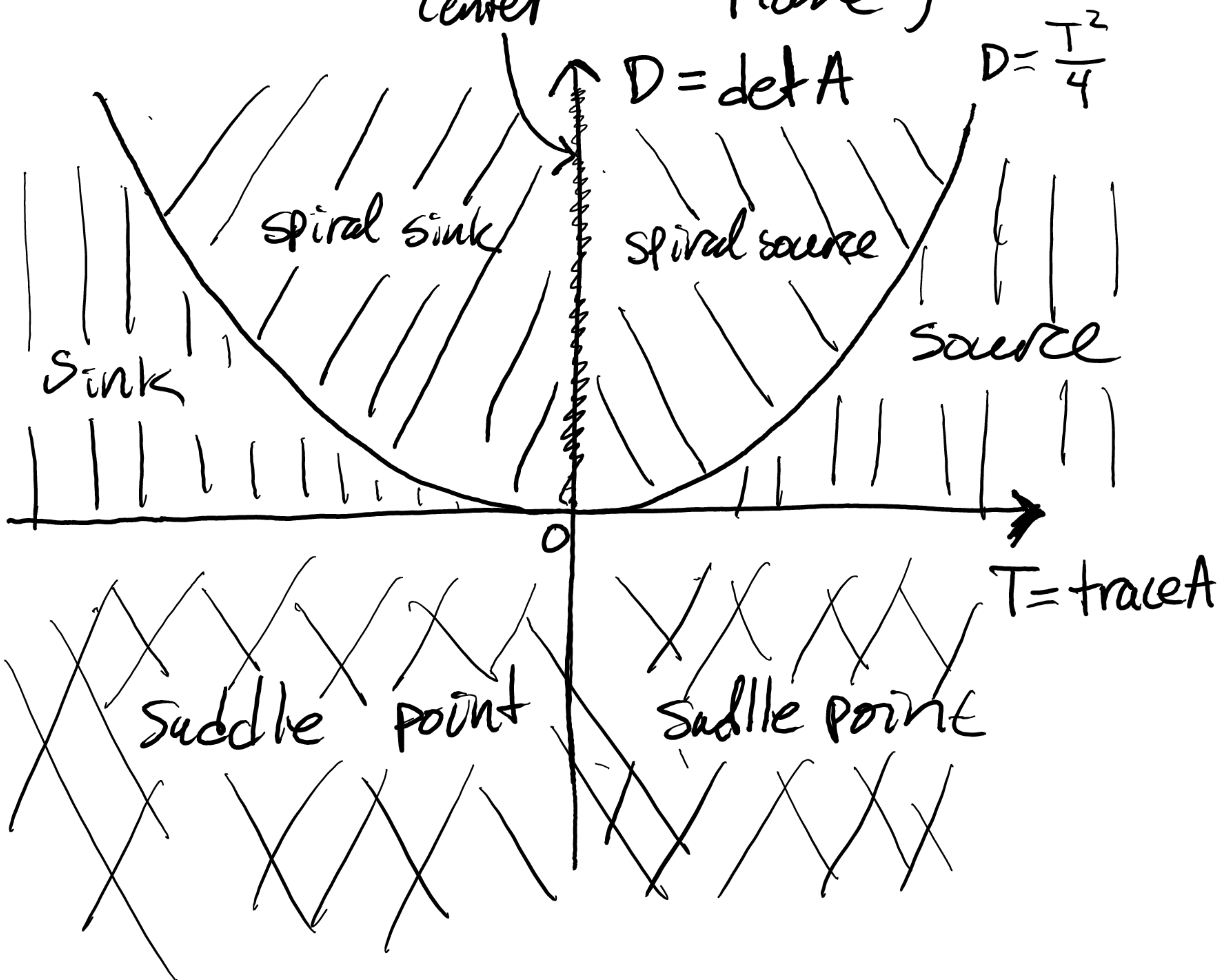
Next, consider $D < 0$:

Then $T^2 - 4D > T^2$, so

λ_+, λ_- are distinct real eigenvalues
with $\lambda_+ > 0 > \lambda_-$

EPD is a saddle.

EP 0 for $\dot{X} = AX$ (trace-determinant plane)



Exercises

Ex 3.1

Extra part a) Find general solution to $\bar{X}' = A\bar{X}$

$$a) A = \begin{bmatrix} 3 & 5 \\ -2 & -2 \end{bmatrix}$$

char eq. $(3-\lambda)(-2-\lambda) + 10 = 0$

becomes $\lambda^2 - \lambda + 4 = 0$

$$\lambda_{\pm} = \frac{1 \pm \sqrt{1-16}}{2} = \frac{1}{2} \pm i \frac{\sqrt{15}}{2}$$

$(A - \lambda_+ I)V = 0$ becomes

$$\begin{bmatrix} 5/2 - i\frac{\sqrt{15}}{2} & 5 \\ -2 & -\frac{5}{2} - i\frac{\sqrt{15}}{2} \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$V = \begin{bmatrix} 5 \\ -\frac{5}{2} + i\frac{\sqrt{15}}{2} \end{bmatrix} = \begin{bmatrix} 5 \\ -\frac{5}{2} \end{bmatrix}_{v_1} + i \begin{bmatrix} 0 \\ \frac{\sqrt{15}}{2} \end{bmatrix}_{v_2}$$

Sol $\underline{x}(t) = \begin{bmatrix} V_1 & V_2 \end{bmatrix} \underline{I} = \gamma_1(t) V_1 + \gamma_2(t) V_2$

where $\underline{I}$ solves

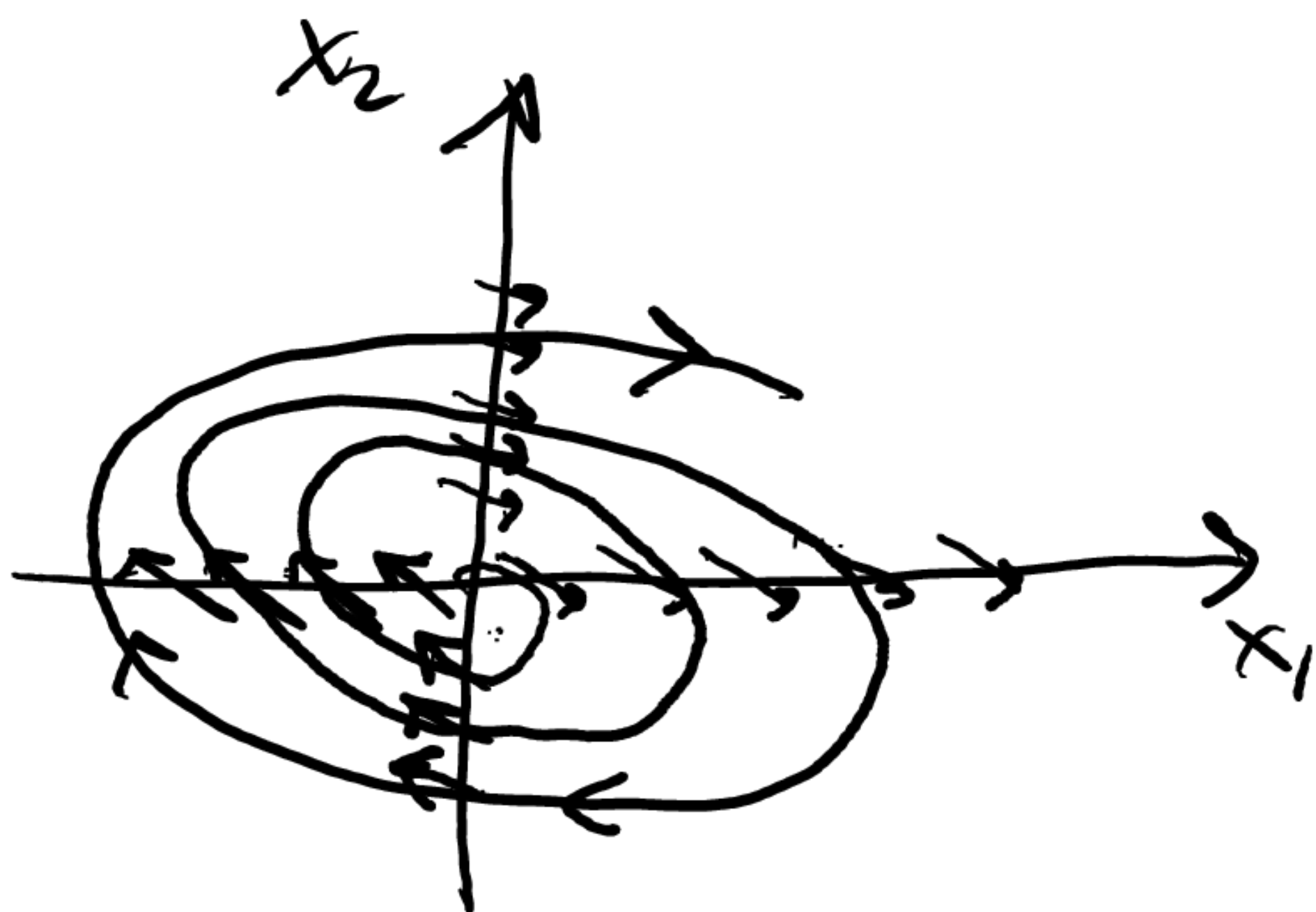
$$\underline{I}' = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} \underline{I}$$

$$\underline{I}(t) = C_1 e^{\frac{t}{2}} \begin{bmatrix} \cos \beta t \\ -\sin \beta t \end{bmatrix} + C_2 e^{\frac{t}{2}} \begin{bmatrix} \sin \beta t \\ \cos \beta t \end{bmatrix}$$

$\alpha > 0 \Rightarrow 0$ is a spiral source

$$F(\underline{x}) = \begin{bmatrix} 3x_1 + 5x_2 \\ -2x_1 - 2x_2 \end{bmatrix}$$

Direction field



Matches 3.

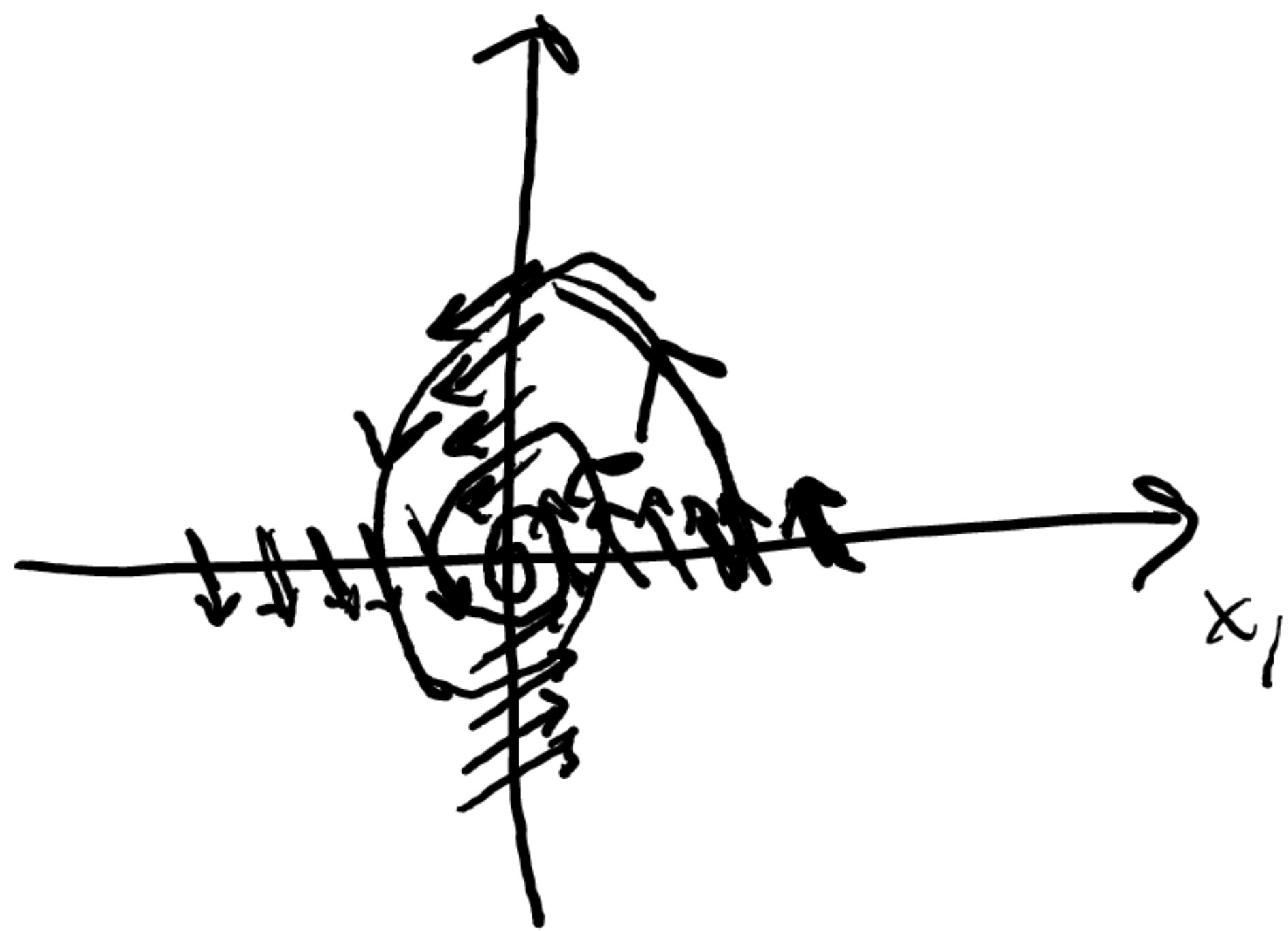
$$b) A = \begin{bmatrix} -3 & -2 \\ 5 & 2 \end{bmatrix},$$

$$\text{Det } A = 4 > 0, \text{ Trace } A = -1 < 0.$$

Trace-determinant classification: O is a spiral sink.

$$\text{Vector field } F(x) = \begin{bmatrix} -3x_1 - 2x_2 \\ 5x_1 + 2x_2 \end{bmatrix}$$

Direction field



Matches 5.

$$c) A = \begin{bmatrix} 3 & -2 \\ 5 & -2 \end{bmatrix}$$

$$\text{Det } A = 4 > 0, \text{ Trace } A = 1 > 0$$

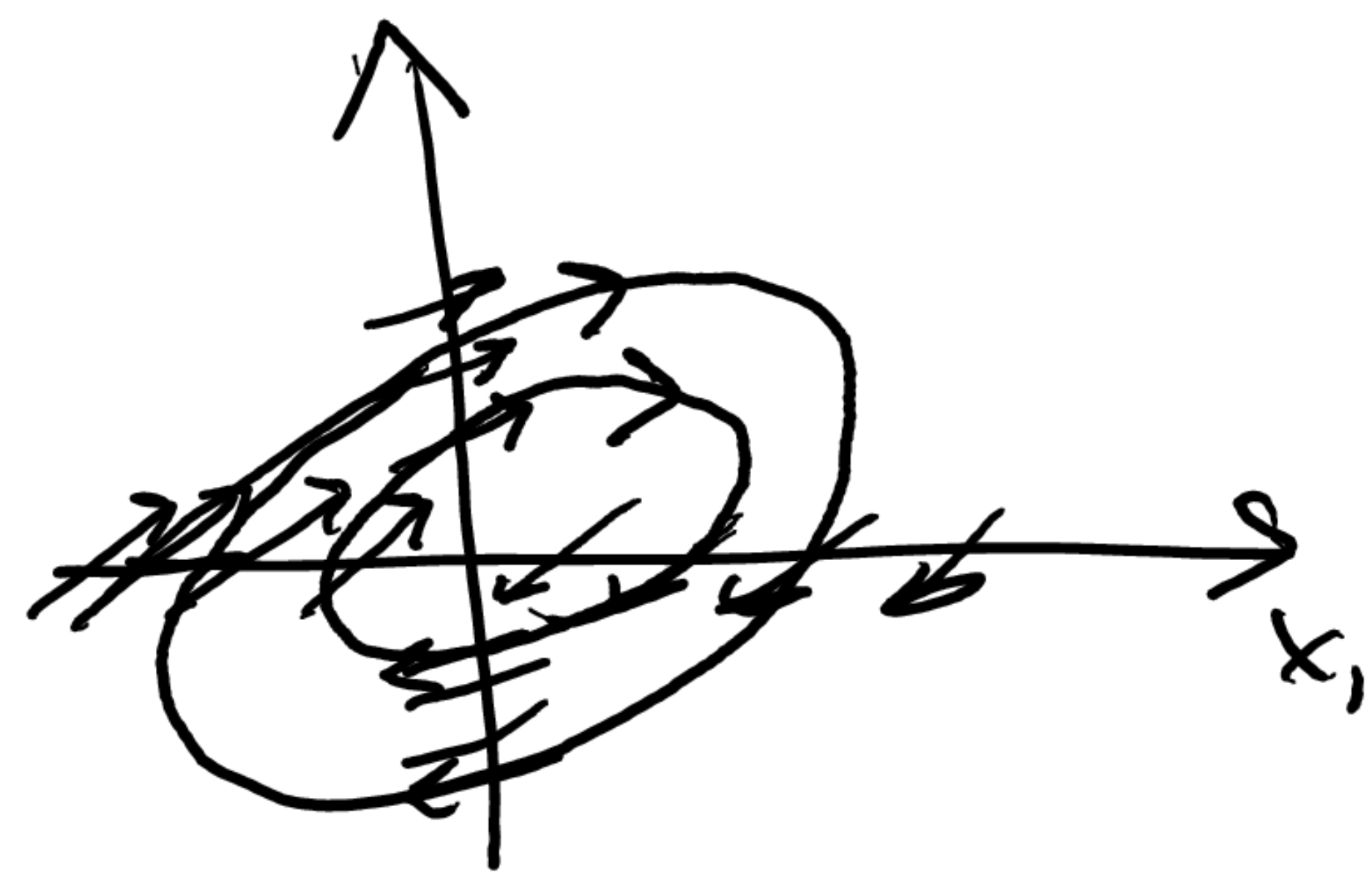
Trace-det classific: O is a spiral source

Matches 6. (only remaining spiral source)

$$d) A = \begin{bmatrix} -3 & 5 \\ -2 & 3 \end{bmatrix}, \quad \det A = 1 > 0 \\ \text{trace } A = 0$$

Trace - det class.: 0 is center

$$F(\underline{x}) = \begin{bmatrix} -3x_1 + 5x_2 \\ -2x_1 + 3x_2 \end{bmatrix}$$



Matches 1.

3.3 a) Find general solution to

$$x'' + x' + x = 0$$

$$\begin{aligned} x_1 &= x \\ x_2 &= x' \end{aligned} \Rightarrow \underline{x}' = \begin{bmatrix} x_2 \\ -x_2 - x_1 \end{bmatrix} = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & -1 \end{bmatrix}}_A \underline{x}$$

Char eq. $\lambda(\lambda + 1) + 1 = 0$

$$\lambda_{\pm} = \underbrace{\frac{-1}{2}}_{\alpha} \pm i \underbrace{\frac{\sqrt{3}}{2}}_{\beta}$$

$(A - \lambda_{+} I) V = 0$ becomes

$$\begin{bmatrix} \frac{1}{2} - i\frac{\sqrt{3}}{2} & 1 \\ -1 & -\frac{1}{2} - i\frac{\sqrt{3}}{2} \end{bmatrix} V = 0$$

with a solution

$$V = \begin{bmatrix} 1 \\ \frac{1}{2} - i\frac{\sqrt{3}}{2} \end{bmatrix} = \underbrace{\begin{bmatrix} 1 \\ \frac{1}{2} \end{bmatrix}}_{V_1} + i \underbrace{\begin{bmatrix} 0 \\ \frac{\sqrt{3}}{2} \end{bmatrix}}_{V_2}$$

Solution of $Y' = \begin{bmatrix} -\frac{1}{2} & \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} & -\frac{1}{2} \end{bmatrix} Y$:

$$Y(t) = c_1 e^{-t/2} \begin{bmatrix} \cos \frac{\sqrt{3}}{2} t \\ -\sin \frac{\sqrt{3}}{2} t \end{bmatrix} + c_2 e^{-t/2} \begin{bmatrix} \sin \frac{\sqrt{3}}{2} t \\ \cos \frac{\sqrt{3}}{2} t \end{bmatrix}$$

$$X(t) = Y_1(t) V_1 + Y_2(t) V_2$$

$$= \begin{bmatrix} Y_1 \\ \frac{Y_1}{2} + \frac{\sqrt{3}}{2} Y_2 \end{bmatrix} \text{ for } c_1, c_2 \in \mathbb{R}.$$

General sol:

$$X(t) = X_1 = Y_1 = c_1 e^{-t/2} \cos \frac{\sqrt{3}}{2} t + c_2 e^{-t/2} \sin \frac{\sqrt{3}}{2} t.$$

3.5 Consider

$$\dot{X} = \underbrace{\begin{bmatrix} a & 1 \\ 2a & 2 \end{bmatrix}}_A X$$

For which points $a \in \mathbb{R}$ do you find a bifurcation.

Char eq. $(a-\lambda)(2-\lambda) - 2a = 0$

$$\lambda^2 - (a+2)\lambda = 0$$

$$\lambda_1 = 0, \lambda_2 = a+2$$

$$V_1 = \begin{bmatrix} 1 \\ -a \end{bmatrix}, V_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix} \quad \text{linearly independent} \\ \Leftrightarrow a \neq -2$$

$$\text{Solution } X(t) = C_1 \begin{bmatrix} 1 \\ -a \end{bmatrix} + C_2 e^{(a+2)t} \begin{bmatrix} 1 \\ 2 \end{bmatrix}$$

See ODE is stable (but not A-stable) for $a < -2$ and unstable for $a > -2$.

What happens at $a = -2$?

$$A = \begin{bmatrix} -2 & 1 \\ -4 & 2 \end{bmatrix}, \quad \lambda := \lambda_1 = \lambda_2 = 0 \text{ and } \dim(\ker(A - \lambda I)) = 1.$$

Find generalized ev: $(A - \lambda I) V_2 = \begin{bmatrix} 1 \\ 2 \end{bmatrix}$
 V_1

$$V_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \quad P = [V_1 \ V_2]$$

$$B = P^{-1} A P = \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$$

$$\dot{\gamma} = B \gamma \Rightarrow \begin{cases} \gamma_1' = \gamma_2 \\ \gamma_2' = 0 \end{cases} \Rightarrow \begin{cases} \gamma_1(t) = c_2 t + c_1 \\ \gamma_2(t) = c_2 \end{cases}$$

has general solution

$$\begin{aligned} x(t) &= P \gamma = (c_1 + c_2 t) V_1 + c_2 V_2 \\ &= \begin{bmatrix} c_1 + c_2 t \\ c_2(2t + 1) + c_1 \end{bmatrix} \end{aligned}$$

ODE is unstable for $a = 0$.

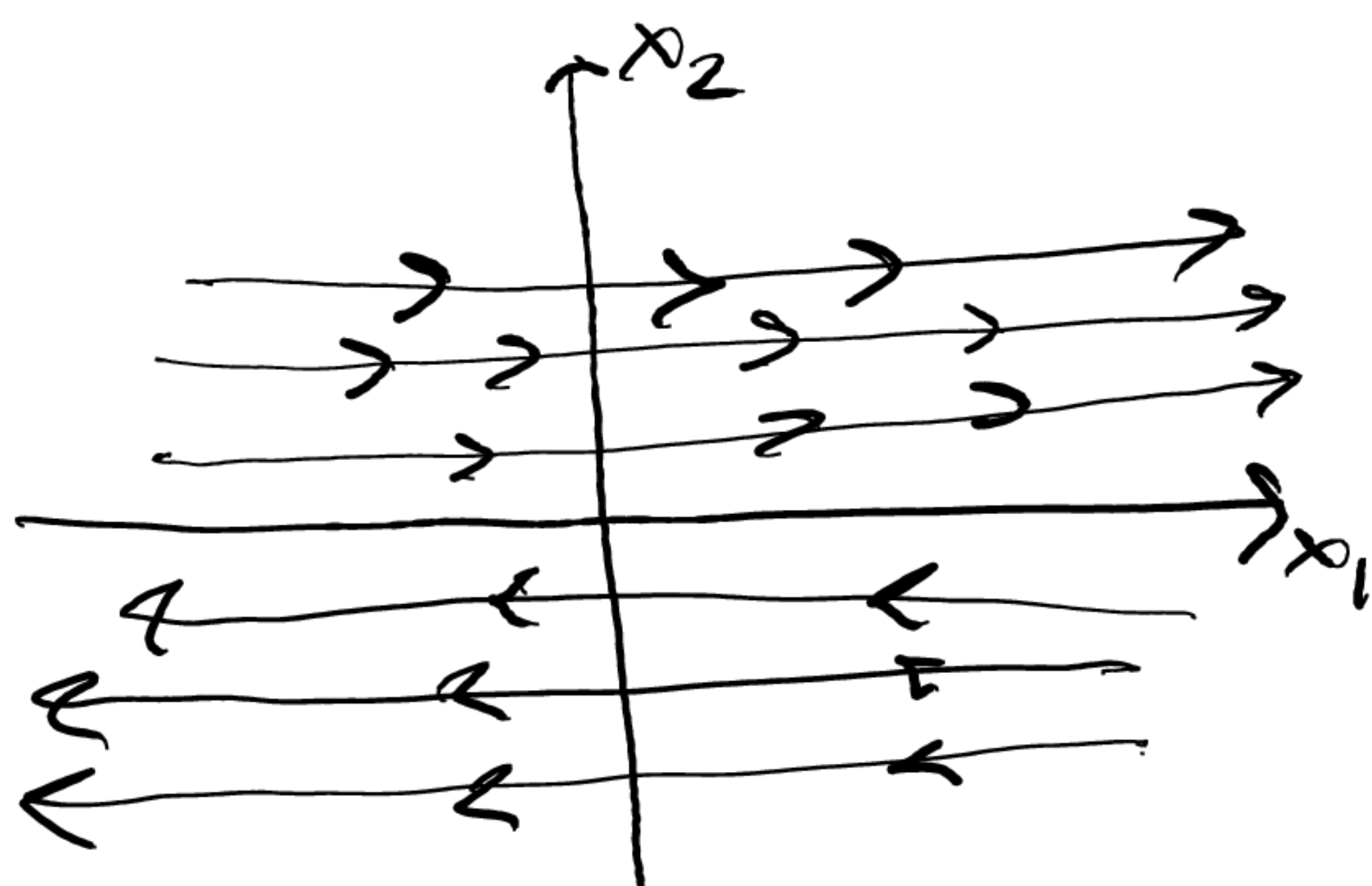
3.11 Find the general solution to
 $\underline{x}' = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \underline{x} = \begin{bmatrix} x_2 \\ 0 \end{bmatrix} = F(\underline{x})$ and

sketch the phase portrait $\begin{matrix} x_1' = x_2 \\ x_2' = 0 \end{matrix}$

Solution: As in 3.5

$$\underline{x}(t) = \begin{bmatrix} c_1 + c_2 t \\ c_1 \end{bmatrix}$$

Phase



3.16 Consider nonlinear system

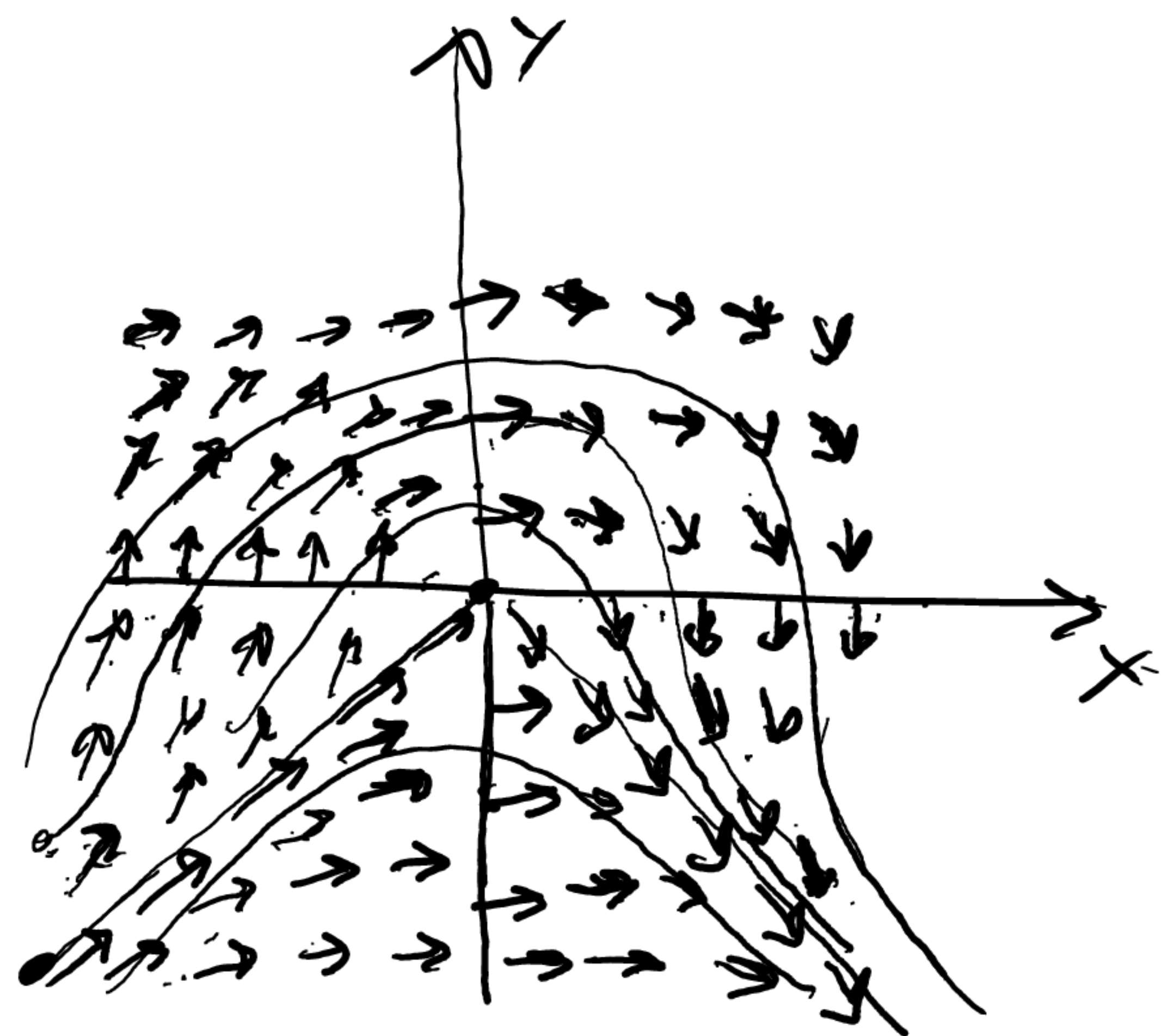
$$x' = |y|$$

$$y' = -x$$

Describe the phase portrait.

Direction field

$$F\left(\begin{bmatrix} x \\ y \end{bmatrix}\right) = \begin{bmatrix} 1-y \\ -x \end{bmatrix} = F\left(\begin{bmatrix} x \\ -y \end{bmatrix}\right)$$



Note: for $t > 0$

$$F\left(\begin{bmatrix} t \\ -t \end{bmatrix}\right) = \begin{bmatrix} t \\ t \end{bmatrix} \perp \begin{bmatrix} 1 \\ 1 \end{bmatrix}$$

$$\& F\left(\begin{bmatrix} t \\ t \end{bmatrix}\right) = \begin{bmatrix} t \\ -t \end{bmatrix} \perp \begin{bmatrix} 1 \\ -1 \end{bmatrix}$$

And for all solutions with $x(0) > 0$,
eventually $y(t) < 0$ and $x(t) \geq x(0) > 0$.

Then $y(s) < 0$ for all $s > t$ & $x(s) > 0 \forall s > t$,

$$\text{so } x'(s) = -y(s)$$

$$y'(s) = -x(s)$$

$$\Rightarrow h(s) = \frac{x}{y} \text{ satisfies } h(t) < 0$$

$$h' = \frac{x'y - y'x}{y^2} = \frac{-y^2 + x^2}{y^2} = h^2 - 1$$

with A-stable EP -1



$$\Rightarrow \lim_{t \rightarrow \infty} h(t) = \lim_{t \rightarrow \infty} \frac{x(t)}{y(t)} = -1$$

(so eventually, for large s , solution $(x(s), y(s))$ is arbitrarily close to line $(s, -s)_{s > 0}$).