

Lectures 9&10 MAT 3440 16/2 & 18/2

Have classified properties of EPO for
 $\mathbf{x}' = A\mathbf{x}$ in terms of eigenpairs of
 $A \in \mathbb{R}^{2 \times 2}$.

Q: When can we more broadly say
that solution paths are
"equivalent"?

For $n \geq 2$ and $A, \hat{A} \in \mathbb{R}^{n \times n}$,
consider ODEs
 $\mathbf{x}' = A\mathbf{x}$, $\hat{\mathbf{x}}' = \hat{A}\hat{\mathbf{x}}$ with
associated flows

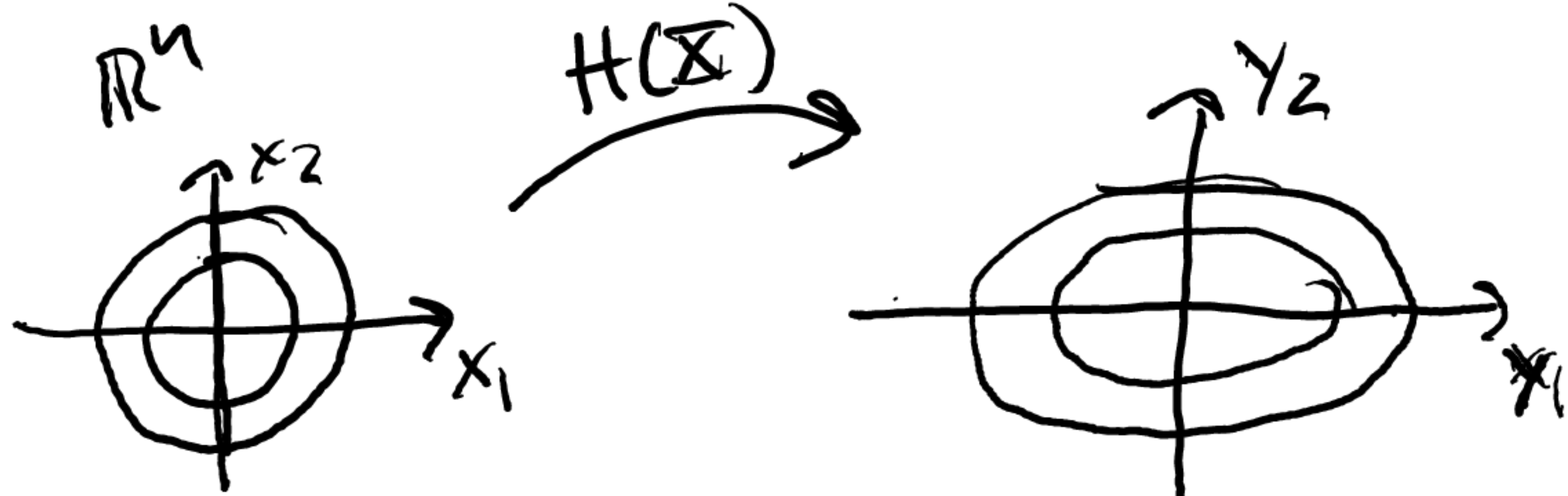
$$\phi_t^A(\mathbf{x}_0) := \mathbf{x}(t; \mathbf{x}(0) = \mathbf{x}_0), \quad \phi_t^{\hat{A}}(\mathbf{x}_0) := \hat{\mathbf{x}}(t; \hat{\mathbf{x}}(0) = \mathbf{x}_0)$$

Recall a mapping $h \in C(\mathbb{R}^n)$ that has a continuous inverse $h^{-1} \in C(\mathbb{R}^n)$ is called a homeomorphism. (so h must be onto: $h(\mathbb{R}^n) = \mathbb{R}^n$, and 1-1: $h(X) = h(Y) \Leftrightarrow X = Y$).

If $h, h^{-1} \in C^k(\mathbb{R}^n)$, $k \geq 1$, then h is called a (C^k) diffeomorphism.

Ex For any invertible $P \in \mathbb{R}^{n \times n}$,
 $h(X) := PX$ with $h(Y) = P^{-1}Y$,
 is a diffeomorphism on $\mathbb{R}^n$.

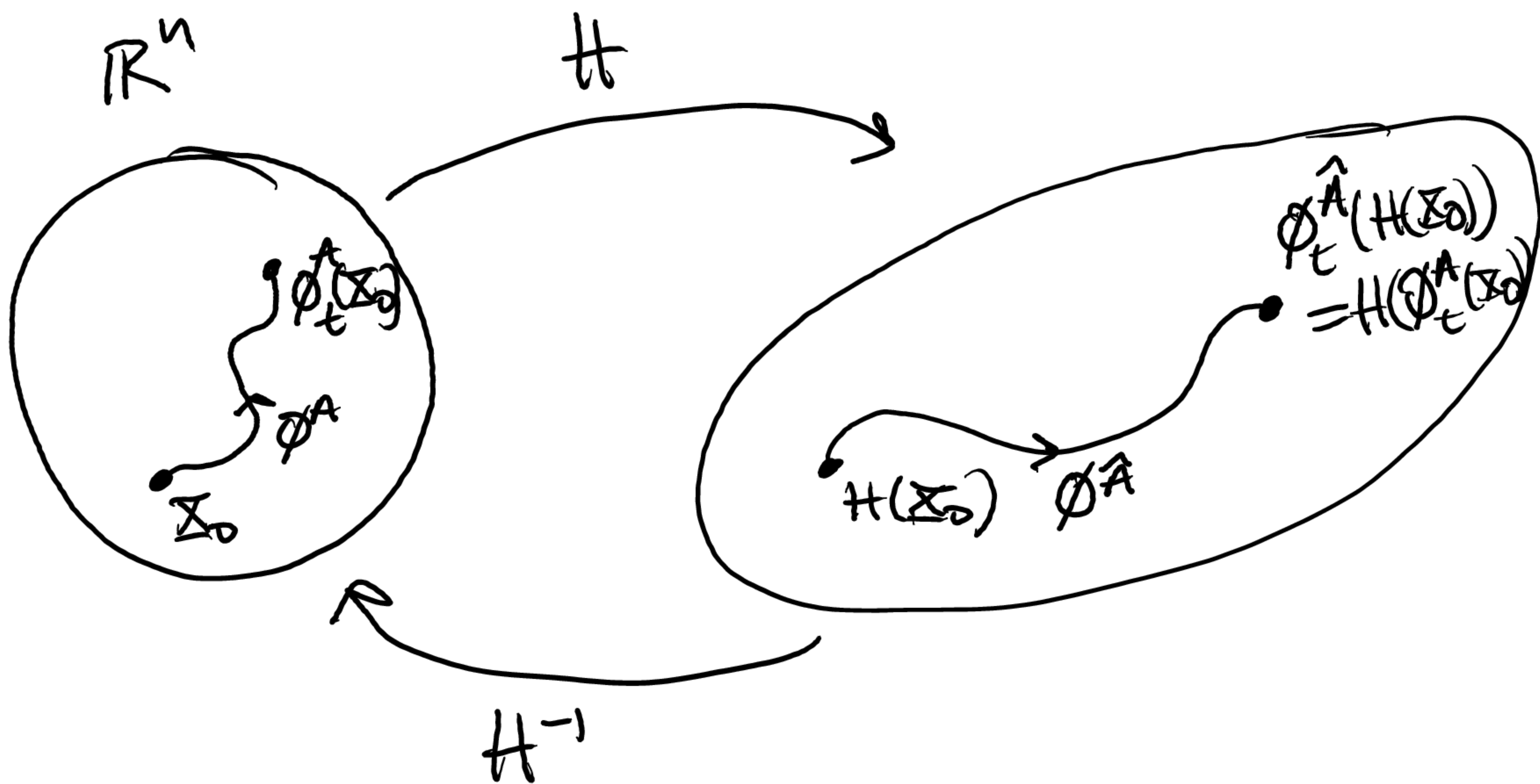
Say $P = \begin{bmatrix} 2 & 0 \\ 0 & 1 \end{bmatrix}$:



Def The flows/ODEs $\Sigma' = A\Sigma$ and $\hat{\Sigma}' = \hat{A}\hat{\Sigma}$ are conjugate if there exists a homeomorphism $H: \mathbb{R}^n \rightarrow \mathbb{R}^n$ s.t.

$$\underbrace{\phi_t^{\hat{A}}(H(\Sigma_0))}_{\hat{\Sigma}(t; \hat{\Sigma}(0)=H(\Sigma_0))} = \underbrace{H(\phi_t^A(\Sigma_0))}_{H(\Sigma(t; \Sigma(0)=\Sigma_0))} \quad \forall t, \Sigma_0.$$

$$\hat{\Sigma}(t; \hat{\Sigma}(0)=H(\Sigma_0)) \quad H(\Sigma(t; \Sigma(0)=\Sigma_0))$$



Lemma 1 For $\Sigma_i' = A_i \Sigma_i$, $A_i \in \mathbb{R}^{n \times n}$, let $\Sigma_i \sim \Sigma_j$ denote that Σ_i is conjugate with Σ_j . Conjugate is an equivalence relation.

$$a) \underline{x}_i \sim \underline{x}_i \quad (\text{reflexive})$$

$$b) \underline{x}_i \sim \underline{x}_j \Leftrightarrow \underline{x}_j \sim \underline{x}_i \quad (\text{symmetry})$$

$$c) \underline{x}_i \sim \underline{x}_j, \underline{x}_j \sim \underline{x}_k \Rightarrow \underline{x}_i \sim \underline{x}_k \quad (\text{transitivity})$$

Proof a) $H(\underline{x}) = \underline{x} \Rightarrow \phi_t^{A_i}(H(\underline{x}_0)) = H(\phi_t^{A_i}(\underline{x}_0)).$

b) If $\phi_t^{A_j}(H(\underline{x}_0)) = H(\phi_t^{A_j}(\underline{x}_0)) \quad \forall t \in \mathbb{R}, \underline{x}_0 \in \mathbb{R}^n$

then writing $\underline{y}_0 = H(\underline{x}_0)$

$$H^{-1}(\phi_t^{A_j}(\underline{y}_0)) = \phi_t^{A_i}(H^{-1}(\underline{y}_0)) \quad \forall t \in \mathbb{R}, \underline{y}_0 \in H(\mathbb{R}^n) = \mathbb{R}^n$$

and H homeomorphism $\Leftrightarrow H^{-1}$ homeomorphism.

c) Let $H_j, H_k : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be homeomorphisms

s.t.

$$\phi_t^{A_j}(H_j(\underline{x}_0)) = H_j(\phi_t^{A_i}(\underline{x}_0)) \quad (\underline{x}_i \sim \underline{x}_j)$$

$$\phi_t^{A_k}(H_k(\underline{x}_0)) = H_k(\phi_t^{A_j}(\underline{x}_0)) \quad (\underline{x}_j \sim \underline{x}_k)$$

Then $H = H_k \circ H_j$ is a homeomorphism and

for $\underline{y}_0 = H_j^{-1}(\underline{y}_0)$

$$\begin{aligned} \phi_t^{A_k}(H(\underline{x}_0)) &= \phi_t^{A_k}(H_k(\underline{x}_0)) = H_k(\phi_t^{A_j}(\underline{x}_0)) \\ &= H_k(\phi_t^{A_j}(H_j(\underline{y}_0))) = H_k \circ H_j(\phi_t^{A_i}(\underline{y}_0)) \end{aligned}$$

If the ODEs $\underline{X}' = A\underline{X}$ and $\hat{\underline{X}}' = \hat{A}\hat{\underline{X}}$ are conjugate with homeomorphism H , then

- $\underline{X}^*$ is an EP of $\underline{X}' = A\underline{X} \Leftrightarrow H(\underline{X}^*)$ is an EP of $\hat{\underline{X}}' = \hat{A}\hat{\underline{X}}$
(EPs are preserved 1-1, and number of sinks/sources often preserved, see next theorem)
- Periodic orbits of ϕ_t^A correspond to periodic orbits of $\phi_t^{\hat{A}}$

Planar systems

For any $A \in \mathbb{R}^{2 \times 2}$, there exists an P s.t.

$$B = P^{-1}AP \text{ equals } \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \text{ or } \begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix} \text{ or } \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}.$$

Lemma 2 $\underline{X}' = A\underline{X}$ is conjugate to $\underline{Y}' = B\underline{Y}$.

Proof $\underline{X} = P\underline{Y} \Rightarrow \underline{Y}(0) = P^{-1}\underline{X}(0)$ and

$$\phi_t^A(\underline{X}_0) = \underline{X}(t; \underline{X}_0) = P\underline{Y}(t; P^{-1}\underline{X}_0) = P\phi_t^B(P^{-1}\underline{X}_0)$$

$$\Rightarrow P^{-1}\phi_t^A(\underline{X}_0) = \phi_t^B(P^{-1}\underline{X}_0) \quad (H(t) = P^{-1})$$

A square matrix A is called hyperbolic if $\operatorname{Re}(\lambda) \neq 0$ for all of its eigenvalues.

Theorem Suppose $A, \hat{A} \in \mathbb{R}^{2 \times 2}$ are hyperbolic. Then $X' = AX$ and $\hat{X}' = \hat{A}\hat{X}$ are conjugate iff A and $\hat{A}$ have the same number of eigenvalues with negative real part.

NB! Means that spiral sinks  and sinks  are conjugate. Same for sources.

Proof sketch

If: (Only for Case I) Suppose A is similar to

$$B = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \text{and} \quad \hat{A} \text{ to } \hat{B} = \begin{bmatrix} \hat{\lambda}_1 & 0 \\ 0 & \hat{\lambda}_2 \end{bmatrix} \quad \text{with}$$
$$\lambda_1 < \lambda_2 \quad \text{and} \quad \hat{\lambda}_1 < \hat{\lambda}_2 \quad \text{real and}$$

and $\lambda_1 \hat{\lambda}_1 > 0$, $\lambda_2 \hat{\lambda}_2 > 0$ (same number of negative EVs).

By Lemma 2, $\underline{X} \sim \underline{Y}$ where $\underline{Y}' = \underline{B}\underline{Y}$, and $\hat{\underline{X}} \sim \hat{\underline{Y}}$ where $\hat{\underline{Y}}' = \hat{\underline{B}}\hat{\underline{Y}}$, with respective solutions

$$\phi_t^{\underline{B}}(\underline{Y}) = \begin{bmatrix} y_1 e^{\lambda_1 t} \\ y_2 e^{\lambda_2 t} \end{bmatrix}, \quad \phi_t^{\hat{\underline{B}}}(\hat{\underline{Y}}) = \begin{bmatrix} \hat{y}_1 e^{\hat{\lambda}_1 t} \\ \hat{y}_2 e^{\hat{\lambda}_2 t} \end{bmatrix}$$

Take $h_i(y) = \text{sign}(y) |y|^{\hat{\lambda}_i / \lambda_i}$ $y \in \mathbb{R}$, ($i=1,2$)
(h_i continuous, strictly increasing and onto so a homeomorphism on $\mathbb{R}$) and set

$$H(\underline{Y}) = \begin{bmatrix} h_1(y_1) \\ h_2(y_2) \end{bmatrix} \quad (\text{a homeomorphism on } \mathbb{R}^2)$$

$$\begin{aligned} \text{Then } \phi_t^{\hat{\underline{B}}}(\hat{\underline{Y}}) &= \begin{bmatrix} \text{sign}(y_1) |y_1|^{\hat{\lambda}_1 / \lambda_1} e^{\lambda_1 t} \\ \text{sign}(y_2) |y_2|^{\hat{\lambda}_2 / \lambda_2} e^{\lambda_2 t} \end{bmatrix} \\ &= H(\phi_t^{\underline{B}}(\underline{Y})) \quad \forall \left(\begin{array}{l} \underline{Y} \in \mathbb{R}^2 \\ t \in \mathbb{R} \end{array} \right) \end{aligned}$$

Hence, $\underline{X} \sim \underline{Y} \sim \hat{\underline{Y}} \sim \hat{\underline{X}} \xrightarrow{\text{Lemma 1c}} \underline{X} \sim \hat{\underline{X}}$.

Only if

Suppose $\text{Re}(\lambda_i) < 0$ $i=1,2$
while $\text{Re}(\hat{\lambda}_2) > 0$.

Then $|\hat{\underline{Y}}(t)| \rightarrow \infty$ as $t \rightarrow \infty$ for $\hat{Y}_2(0) \neq 0$
 while $|\underline{Y}(t)| \rightarrow 0$.

For an arbitrary homeomorphism $H: \mathbb{R}^2 \rightarrow \mathbb{R}^2$,
 take $\underline{Y}_0 = H^{-1}(\begin{bmatrix} 0 \\ 1 \end{bmatrix})$. Then

$$|\phi_t^{\hat{B}}(H(\underline{Y}_0))| = |\phi_t^{\hat{B}}(\begin{bmatrix} 0 \\ 1 \end{bmatrix})| \rightarrow \infty \text{ as } t \rightarrow \infty$$

while

$$|H(\phi_t^B(\underline{Y}_0))| = |H(\begin{bmatrix} 0 \\ 0 \end{bmatrix})| \neq \infty, \text{ since}$$

$\phi_t^B(\underline{Y}_0) \rightarrow \begin{bmatrix} 0 \\ 0 \end{bmatrix}$. Hence $\underline{Y}$ is not conjugate
 with $\hat{\underline{Y}}$ since

$$\phi_t^{\hat{B}}(H(\underline{Y}_0)) \neq H(\phi_t^B(\underline{Y}_0)) \quad \forall t \in \mathbb{R}, \underline{Y}_0 \in \mathbb{R}^2$$

does not hold for any H .

Suppose $\underline{X} \sim \hat{\underline{X}}$. Then

$$\underline{Y} \sim \underline{X} \sim \hat{\underline{X}} \sim \hat{\underline{Y}}, \text{ contradicting } \underline{Y} \not\sim \hat{\underline{Y}}.$$

See §4.2 for other cases

~~is~~

4.1 Consider family of ODEs

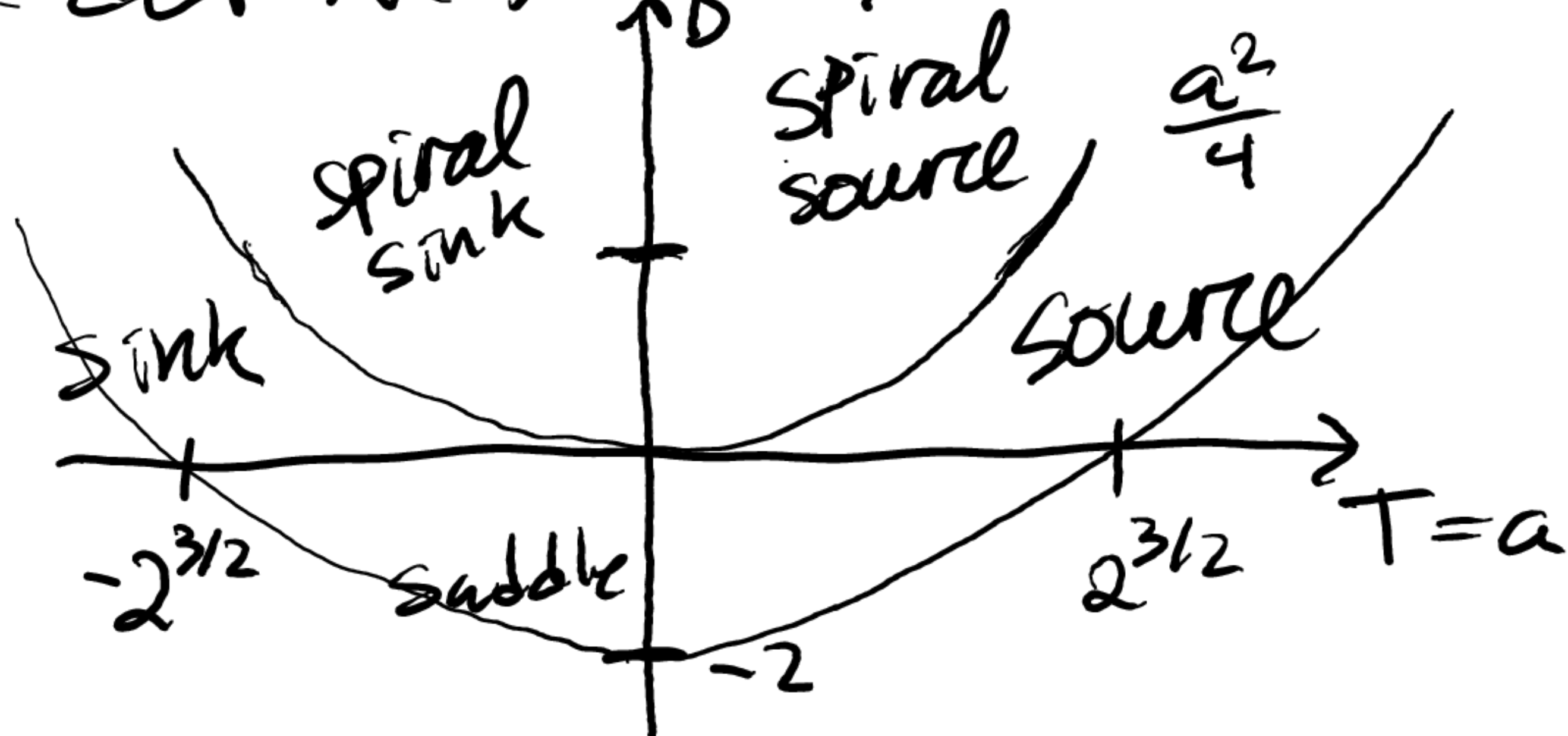
$$\dot{X} = \begin{bmatrix} a & \sqrt{2} + a/2 \\ \sqrt{2} - a/2 & 0 \end{bmatrix} X, \quad a \in \mathbb{R}$$

- Sketch the path taken by the ODE in the trace-determinant plane as a varies
- Discuss any bifurcations that occur along this path and compute corresponding values of a

Sol a) Char eq. $\lambda(\lambda - a) + 2 - \frac{a^2}{4} = 0$

$$T(a) = \text{trace } A(a) = a$$

$$D(a) = \det A(a) = \frac{a^2}{4} - 2$$



b) Bifurcations as a increases:

$a = -2^{\frac{3}{2}}$ from sink to saddle

$a = 2^{\frac{3}{2}}$ from saddle to source

4.2 sketch the analogue of the trace-determinant plane for

$$\underline{x}' = \begin{bmatrix} a & b \\ b & a \end{bmatrix} \underline{x} \quad \text{in the } ab\text{-plane}$$

(In terms of where it has similar phase portraits.)

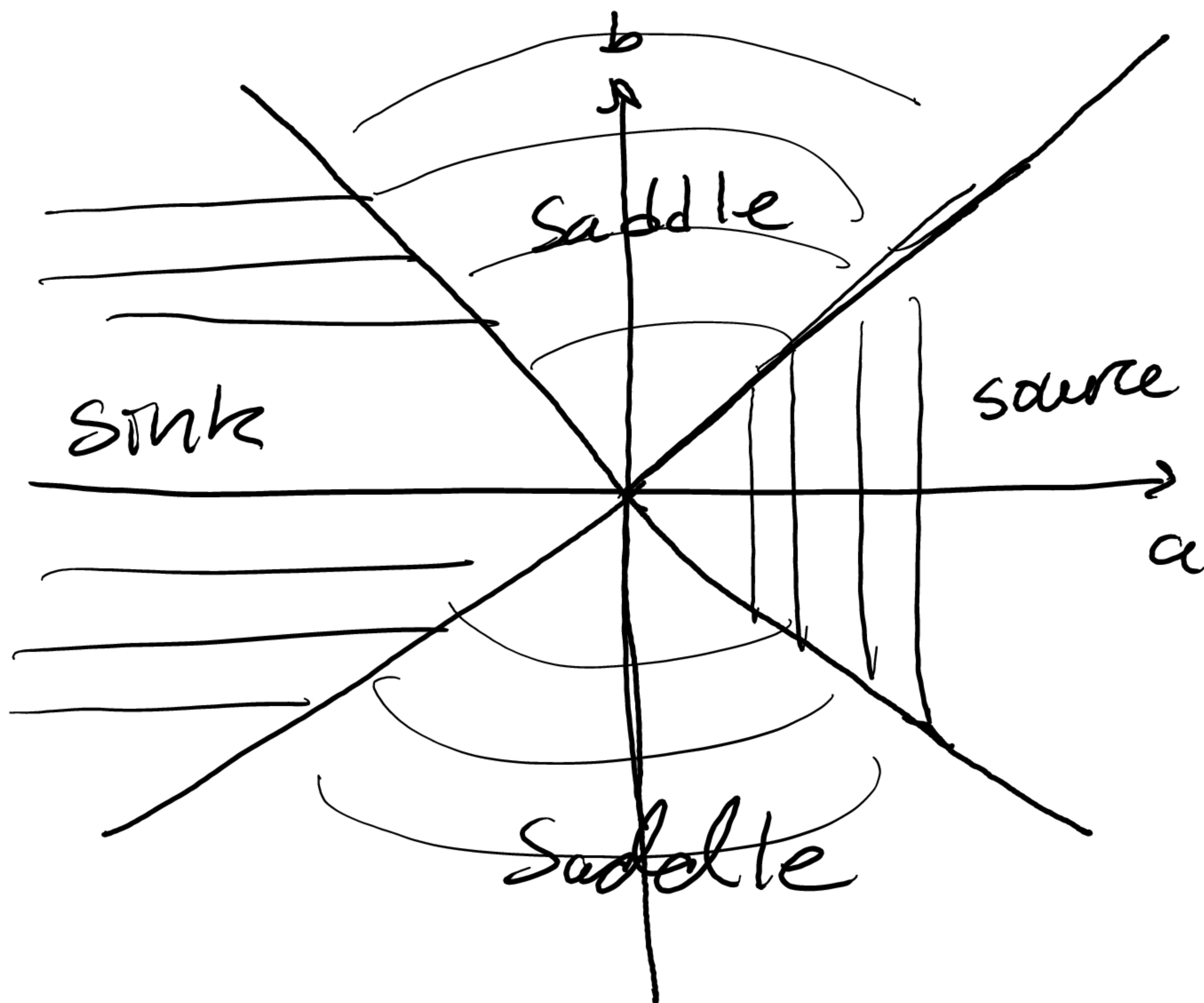
Char eq. $\lambda^2 - 2\lambda a + a^2 - b^2 = 0$

$$\lambda_{\pm}(a,b) = a \pm |b|$$

$\lambda_+ > \lambda_-$. Source $\lambda_- > 0$ when $a > |b|$

Sink $\lambda_+ < 0$ when $a < -|b|$

Saddle $\lambda_+ \lambda_- < 0$ when $a \in (-|b|, |b|)$



Bifurcations happen along lines
 $b(a) = a$ and $b(a) = -a$.

4.4 Prove that $H(x, y) = \begin{bmatrix} x \\ -y \end{bmatrix}$ provides a
conjugacy between

$$\mathbb{X}' = \underbrace{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}_A \mathbb{X} \quad \text{and} \quad \mathbb{Y}' = \underbrace{\begin{bmatrix} 1 & -1 \\ 1 & 1 \end{bmatrix}}_B \mathbb{Y}$$

Sol

Will show that $\phi_t^B(H(\mathbb{Z}_0)) = H(\phi_t^A(\mathbb{Z}_0))$

Note that

$$x_1' = x_1 + x_2$$

$$x_2' = -x_1 + x_2$$

while

$$y_1' = y_1 - y_2$$

$$y_2' = y_1 + y_2$$

Setting $Z = \begin{bmatrix} z_1 \\ z_2 \end{bmatrix} := H(y_1, y_2) = \begin{bmatrix} y_1 \\ -y_2 \end{bmatrix}$, yields

$$\left. \begin{aligned} z_1' &= y_1' = z_1 + z_2 \\ z_2' &= -y_2' = -z_1 + z_2 \end{aligned} \right\} \Rightarrow Z' = \underbrace{\begin{bmatrix} 1 & 1 \\ -1 & 1 \end{bmatrix}}_A Z$$

$$\Rightarrow Z(t; Z_0) = \Phi_t^A(Z_0) = Z(t; Z_0)$$

$$Z(0) = Z_0 \Rightarrow Z(0) = H^{-1}(Z_0), \text{ and}$$

$$\begin{aligned} Z(t; Z_0) &= H(Z(t; H^{-1}(Z_0))) \\ &= H(\Phi_t^B(H^{-1}(Z_0))) \end{aligned}$$

$$\Rightarrow \Phi_t^B(H^{-1}(Z_0)) = H^{-1}(\Phi_t^A(Z_0)).$$

□

4.6 Prove that any two planar ODE

$$\underline{X}' = A \underline{X} \quad , \quad \hat{\underline{X}}' = \hat{A} \hat{\underline{X}} \quad \text{with}$$

non-hyperbolic matrices and same eigenvalues

$$\lambda_{\pm} = \pm i\beta \quad \& \quad \hat{\lambda}_{\pm} = \pm i\beta$$

are conjugate.

What if $\lambda_{\pm} = \pm i\beta$ & $\hat{\lambda}_{\pm} = \pm i\gamma$, $\gamma \neq \beta$?

What if $\gamma = -\beta$?

Sol A is similar to $B = \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix} = P^{-1} A P$

By Lemma 2 in today's lecture,

$H(\underline{X}) = P \underline{X}$ provides a conjugacy between

$\underline{X}' = A \underline{X}$ and $\underline{Y}' = \begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix} \underline{Y}$, namely

$$\phi_t^B(P^{-1} \underline{X}_0) = P^{-1} \phi_t^A(\underline{X}_0)$$

similarly, $\phi_t^B(\hat{P}^{-1} \underline{X}_0) = \hat{P}^{-1} \phi_t^{\hat{A}}(\underline{X}_0)$.

Since conjugacy is an equivalence relation

$$X \sim Y \sim \hat{X} \Rightarrow X \sim \hat{\hat{X}}.$$

If $\hat{\lambda}_{\pm} = i\gamma$, $\gamma \neq \beta$,

$$X \sim \hat{X} \Leftrightarrow Y \sim \hat{Y} \quad \text{where}$$

$$Y' = \underbrace{\begin{bmatrix} 0 & \beta \\ -\beta & 0 \end{bmatrix}}_{\hat{B}} Y \quad \& \quad \hat{Y}' = \underbrace{\begin{bmatrix} 0 & \gamma \\ -\gamma & 0 \end{bmatrix}}_{\hat{\hat{B}}} \hat{Y}.$$

Solutions

$$Y(t; X) = x_1 \begin{bmatrix} \cos \beta t \\ -\sin \beta t \end{bmatrix} + x_2 \begin{bmatrix} \sin \beta t \\ \cos \beta t \end{bmatrix}$$

$$\hat{Y}(t; X) = x_1 \begin{bmatrix} \cos \gamma t \\ -\sin \gamma t \end{bmatrix} + x_2 \begin{bmatrix} \sin \gamma t \\ \cos \gamma t \end{bmatrix}$$

If $|\beta| \neq |\gamma|$, suppose WLOG that $|\beta| < |\gamma|$. Then

Y is $\frac{2\pi}{|\beta|}$ periodic and $\hat{Y}$ is $\frac{2\pi}{|\gamma|}$ periodic, so

for any homeomorphism $H: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ & $X \in \mathbb{R}^2$,

$$H(X) = \phi_{\frac{2\pi}{|\gamma|}}^{\hat{B}}(H(X))$$

while $H(\phi_{\frac{\beta}{\gamma}}^B(X)) \neq H(X)$ for

instance for $X = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, as

$$\text{as then } \phi_{\frac{\beta}{\gamma}}^B \left(\begin{bmatrix} 1 \\ 0 \end{bmatrix} \right) = \begin{bmatrix} \cos \frac{2\pi\beta}{\gamma} \\ -\sin \frac{2\pi\beta}{\gamma} \end{bmatrix} \neq \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad \text{if } \frac{\beta}{\gamma} \notin \mathbb{Z}.$$

so $H(\phi_{\frac{\beta}{\gamma}}^B(X)) \neq H(X)$ since H is 1-1.

Hence,

$$\phi_{\frac{\beta}{\gamma}}^{\hat{B}}(H(X)) = H(\phi_{\frac{\beta}{\gamma}}^B(X)) \quad \forall X \in \mathbb{R}^2$$

is not true for any homeom. $H \Rightarrow X \times \hat{X}$

If $\beta \neq \gamma$ and β or $\gamma = 0$, suppose $\gamma = 0$, then

$$\phi_t^{\hat{B}}(X) = X \quad \forall t, \text{ and it follows}$$

similarly as above that $I \times \hat{I} \Rightarrow X \times \hat{X}$.

If $\gamma = -\beta$, then

$$\hat{Y}(t; X) = x_1 \begin{bmatrix} \cos \beta t \\ \sin \beta t \end{bmatrix} + x_2 \begin{bmatrix} -\sin \beta t \\ \cos \beta t \end{bmatrix}$$

The HM $H(\mathbb{X}) \simeq \begin{bmatrix} x_2 \\ x_1 \end{bmatrix}$ yields

$$\hat{Y}(t; H(\mathbb{X})) = \begin{bmatrix} x_2 \cos \beta t - x_1 \sin \beta t \\ x_2 \sin \beta t + x_1 \cos \beta t \end{bmatrix}$$

$$\begin{aligned} * H(Y(t; \mathbb{X})) &= H \left(\begin{bmatrix} x_1 \cos \beta t + x_2 \sin \beta t \\ -x_1 \sin \beta t + x_2 \cos \beta t \end{bmatrix} \right) \\ &= \hat{Y}(t; H(\mathbb{X})) \quad \forall t, \mathbb{X} \Rightarrow Y \sim \hat{Y} \end{aligned}$$

$$\left(\text{that is } H(\Phi_t^B(\mathbb{X})) = \Phi_t^{\hat{B}}(H(\mathbb{X})) \right)$$

$$\text{so } \mathbb{X} \sim \hat{\mathbb{X}}.$$



Exercise For scalar ODEs

$$x' = \lambda x \quad \text{and} \quad \hat{x} = \hat{\lambda} x \quad \text{with}$$

$\lambda, \hat{\lambda} \in \mathbb{R} \setminus \{0\}$, show that x and $\hat{x}$ are conjugate iff $\lambda \hat{\lambda} > 0$.

Second, what can you say about conjugacy if $\lambda = 0$ and $\hat{\lambda} \in \mathbb{R}$?

Sol Note $\phi_t^\lambda(x_0) = x_0 e^{\lambda t}$ & $\phi_t^{\hat{\lambda}} = x_0 e^{\hat{\lambda} t}$.

x and $\hat{x}$ are conjugate iff

$$\phi_t^{\hat{\lambda}}(h(x_0)) = h(\phi_t^\lambda(x_0)) \quad \forall t, x_0 \in \mathbb{R}.$$

for some homeomorphism $h: \mathbb{R} \rightarrow \mathbb{R}$.

This is equivalent to

$$h(x_0) e^{\hat{\lambda} t} = h(x_0 e^{\lambda t}) \quad \forall x_0, t \quad (1)$$

If $\hat{\lambda} \lambda > 0$, then

$h(x) = \text{sign}(x) |x|^{\hat{\lambda}/\lambda}$ makes (1) true.

If $\lambda = 0, \hat{\lambda} \neq 0$ (or vice versa), then

(1) becomes $h(x_0) e^{\hat{\lambda} t} = h(x_0) \quad \forall x_0, t$

which is not possible for any x_0 st.
 $h(x_0) \neq 0$, so not possible for any
homeomorphism (since it is onto $\mathbb{R}$)

If $\lambda < 0 < \hat{\lambda}$ (or vice versa),

then (1) becomes

$$h(x_0) e^{\hat{\lambda} t} = h(x_0 e^{\lambda t}) \quad \forall x_0, t$$

This holds iff

$$h(x_0) = \frac{h(x_0 e^{\lambda t})}{e^{\hat{\lambda} t}} \quad \forall x_0, t$$

which cannot hold, since it requires that

$$h(x_0) = \lim_{t \rightarrow \infty} \frac{h(x_0 e^{\lambda t})}{e^{\hat{\lambda} t}} = 0 \quad \forall x_0$$

If $\lambda = \hat{\lambda} = 0$, then $\phi_t^\lambda(x_0) = \phi_t^{\hat{\lambda}}(x_0) = x_0$
and (1) holds for any homeomorphism h .

Conjugate table for $x \sim \hat{x}$

$\lambda \backslash \hat{\lambda}$	< 0	0	> 0
< 0	true	false	false
0	false	true	false
> 0	false	false	true

4.7 Consider all 2×2 linear systems with exactly one eigenvalue equal to 0. Which of these systems are conjugate?

Sol If $A \in \mathbb{R}^{2 \times 2}$ has EV $\lambda_1 = 0$, then $\lambda_2 \in \mathbb{R}$ (as complex EV come in pairs). Since we assume $\lambda_2 \neq 0$, λ_1, λ_2 are distinct, so A is similar to $B = \begin{bmatrix} 0 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

By Lemma 2, $\mathcal{X}$ is conjugate to $\mathcal{Y}$ solving $\mathcal{Y}' = B\mathcal{Y}$ i.e. to $\mathcal{Y}(t; Z) = \begin{bmatrix} z_1 \\ z_2 e^{\lambda_2 t} \end{bmatrix} = \phi_t^B(Z)$.

$\hat{\mathcal{X}}' = \hat{A}\hat{\mathcal{X}}$ with EV $\hat{\lambda}_1 = 0$ and $\hat{\lambda}_2 \neq 0$

is similarly conjugate to

$\hat{\mathcal{Y}}(t; Z) = \begin{bmatrix} z_1 \\ z_2 e^{\hat{\lambda}_2 t} \end{bmatrix} = \phi_t^{\hat{B}}(Z)$ with $\hat{B} = \begin{bmatrix} 0 & 0 \\ 0 & \hat{\lambda}_2 \end{bmatrix}$

For any homeomorphism $h(\mathcal{X}) = \begin{bmatrix} h_1(\mathcal{X}) \\ h_2(\mathcal{X}) \end{bmatrix}$

$$\phi_t^{\hat{B}}(H(z_1, z_2)) = \begin{bmatrix} h_1(z_1, z_2) e^{\hat{\lambda}_2 t} \\ h_2(z_1, z_2) e^{\hat{\lambda}_2 t} \end{bmatrix}$$

while

$$H(\phi_t^B(z_1, z_2)) = \begin{bmatrix} h_1(z_1, z_2 e^{\lambda_2 t}) \\ h_2(z_1, z_2 e^{\lambda_2 t}) \end{bmatrix}$$

These are equal iff

$$\left. \begin{aligned} h_1(z_1, z_2) &= h_1(z_1, z_2 e^{\lambda_2 t}) \\ h_2(z_1, z_2) e^{\hat{\lambda}_2 t} &= h_2(z_1, z_2 e^{\lambda_2 t}) \end{aligned} \right\} \forall \begin{matrix} z \in \mathbb{R}^2 \\ t \in \mathbb{R} \end{matrix} \quad (2)$$

First equation can only hold if

$$h_1(z_1, z_2) = h_1(z_1) \quad (\text{as } h_1 \text{ injective}).$$

and then we can use $h_1(z_1) = z_1$.

Second equation:

- h_2 must depend on z_2 to ensure it is injective on $\mathbb{R}^2$.
- If $\hat{\lambda}_2 \lambda_2 > 0$, then $h_2(z_2) = \text{sign}(z_2) |z_2|^{\hat{\lambda}_2 / \lambda_2}$ ensures (2) holds.

- If $\hat{\lambda}_2 \lambda_2 < 0$, then for fixed $z_1 = 1$,
(2) yields subconstraint

$$h_2(1, z_2) e^{\hat{\lambda}_2 t} = h_2(1, z_2 e^{\lambda_2 t}) \quad \forall z_2, t$$

Take z_2 s.t.

$h_2(1, z_2) \neq 0$, then if $\hat{\lambda}_2 > 0 > \lambda_2$,

$$h_2(1, z_2) = \lim_{t \rightarrow \infty} e^{-\hat{\lambda}_2 t} h_2(1, z_2 e^{\lambda_2 t}) = 0$$

a contradiction (cannot hold for any t).

So $X \sim \hat{X}$ when $\hat{\lambda}_2 > 0 > \lambda_2$.

By symmetry of the relation $\sim$,
it cannot hold when $\lambda_2 > 0 > \hat{\lambda}_2$ either.
