

Lecture 11,

Today: Linear ODE

$$\mathbf{x}' = A \mathbf{x}$$

for $A \in \mathbb{R}^{n \times n}$ $n \geq 2$.

Let $\lambda_1, \dots, \lambda_m \in \mathbb{C}$ denote the distinct EV of A . Then the characteristic eq. can be written

$$(\lambda_1 - \lambda)^{a_1} (\lambda_2 - \lambda)^{a_2} \dots (\lambda_m - \lambda)^{a_m} = 0$$

where $a_1, \dots, a_m \in \mathbb{N}$ with $\sum_{j=1}^m a_j = n$.

a_j is the algebraic multiplicity of λ_j ,

and $g_j := \dim(\ker(A - \lambda_j I))$

is the geometric multiplicity of λ_j ,

where $\ker(A - \lambda_j I) = \{v \in \mathbb{C}^n \mid (A - \lambda_j I)v = 0\}$.

(When $\lambda_j \in \mathbb{R}$ $\dim(\{v \in \mathbb{R}^n \mid (A - \lambda_j I)v = 0\})$
 $= \dim(\{v \in \mathbb{C}^n \mid (A - \lambda_j I)v = 0\})$).

Linear algebra facts

$$1 \leq g_j \leq a_j \quad \text{for } j=1, \dots, m$$

If $g_j = a_j$ for all $j=1, \dots, m$,
then A is called diagonalizable.

Otherwise, if $g_j < a_j$ for some j ,
then A is called defective.

Ex 1 $A = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 2 \end{bmatrix}$ has

$$\text{char eq } (1-\lambda)^2(2-\lambda)=0$$

E_1, E_2 are EV of $\lambda_1=1 \Rightarrow a_1=g_1=2$

E_3 is EV of $\lambda_2=3, a_2=g_2=1$.

Ex 2 $A = \begin{bmatrix} 1 & 1 & 0 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{bmatrix}$ has char eq
 $(1-\lambda)^3=0$

$\lambda_1 = 1$ with $a_1 = 3$ and only one
lin. indep EV: $V_1 = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}$ so $g_1 = 1$

$$\left(\text{As } (A - I)V = 0 \Leftrightarrow \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \\ v_3 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix} \right)$$

A is defective.

Ex 3 For $\beta \neq 0$

$$A = \begin{bmatrix} 0 & \beta & 0 & 0 \\ -\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{bmatrix} \quad \text{has char eq} \quad (\lambda^2 - \beta^2)^2 = 0$$

$$\text{i.e. } (\lambda + i\beta)^2 (\lambda - i\beta)^2 = 0$$

$$\lambda_1 = i\beta \text{ has } a_1 = 2 \text{ \& } V_1 = \begin{bmatrix} 1 \\ i \\ 0 \\ 0 \end{bmatrix} \quad V_2 = \begin{bmatrix} 0 \\ 0 \\ 1 \\ i \end{bmatrix}$$

so $g_1 = 2$

And $\lambda_2 = -i\beta$ with $a_2 = 2$, $g_2 = 2$.

$$\hat{A} = \begin{bmatrix} 0 & \beta & 1 & 0 \\ -\beta & 0 & 0 & 1 \\ 0 & 0 & 0 & \beta \\ 0 & 0 & -\beta & 0 \end{bmatrix}$$

has the same
char eq.

$$\lambda_1 = i\beta, \quad a_1 = 2, \quad \text{but } g_1 = 1 \text{ as}$$

$$(\hat{A} - \lambda_1 I)V = 0 \text{ yields only one}$$

$$\text{lin. indep. eigenvector } V_1 = \begin{bmatrix} 1 \\ i \\ 0 \\ 0 \end{bmatrix}$$

Conjugate pair always has same properties:

$$\lambda_2 = -i\beta \text{ and } a_2 = 2, \quad g_2 = 1.$$

$\hat{A}$ is degenerate.

Jordanization of $A \in \mathbb{R}^{n \times n}$

Consider first three types

II A is diagonalizable and has real distinct eigenvalues $\lambda_1, \dots, \lambda_m \in \mathbb{R}$, $1 \leq m \leq n$ with $\sum_{j=1}^m a_j = n$ and EV $v_1, \dots, v_n$.

Then $P = [v_1 \dots v_n]$ is invertible and

$$B = P^{-1}AP = \begin{bmatrix} \Lambda_1 & & \\ & \Lambda_2 & \\ & & \ddots \\ & & & \Lambda_m \end{bmatrix} \in \mathbb{R}^{n \times n}$$

$$\text{where } \Lambda_j = \begin{bmatrix} \lambda_j & & \\ & \ddots & \\ & & \lambda_j \end{bmatrix} \in \mathbb{R}^{a_j \times a_j}$$

Proof For convenience, relabel eigenvalues $\lambda_1, \dots, \lambda_n$, possibly non-distinct. Then

$$B = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} \text{ and } P^{-1}AP = P^{-1}[Av_1 \dots Av_n]$$

$$= P^{-1} [\lambda_1 v_1 \quad \dots \quad \lambda_n v_n]$$

$$= [\lambda_1 P^{-1} v_1 \quad \dots \quad \lambda_n P^{-1} v_n] = [\lambda_1 E_1 \quad \dots \quad \lambda_n E_n]$$

$\Sigma' = A\Sigma$ has related ODE $\Sigma' = B\Sigma$ i.e.

$$\left. \begin{array}{l} y_1' = \lambda_1 y_1 \\ \vdots \\ y_n' = \lambda_n y_n \end{array} \right\} \Rightarrow \left. \begin{array}{l} y_1 = c_1 e^{\lambda_1 t} \\ \vdots \\ y_n = c_n e^{\lambda_n t} \end{array} \right\} =: \Sigma(t; C) \quad C \in \mathbb{R}^n$$

General solution Σ :

$$\Sigma(t) = P \Sigma = \sum_{j=1}^n c_j e^{\lambda_j t} v_j$$

Case $\boxed{2}$ If A is diagonalizable, but has complex eigen values, then we order the eigenvalues in (possibly repeating) real ones: $(\lambda_1, V_1), \dots, (\lambda_k, V_k)$

thereafter $l = \frac{n-k}{2}$ complex pairs

$$\{(\sigma_1 = \alpha_1 + i\beta_1, U_1), (\bar{\sigma}_1, \bar{U}_1)\}$$

$$\{(\sigma_l, U_l), (\bar{\sigma}_l, \bar{U}_l)\} \text{ - Columns } V_j \text{ in } P:$$

$$\text{We set } V_{k+2j-1} = \text{Re}(U_j), V_{k+2j} = \text{Im}(U_j)$$

$$\text{for } j = 1, \dots, l. \text{ Then } P = [V_1 \dots V_n]$$

is invertible and

$$B = P^{-1}AP = \begin{bmatrix} \lambda_1 & & & \\ & \ddots & & \\ & & \lambda_k & \\ & & & \Delta_1 & & \\ & & & & \ddots & \\ & & & & & \Delta_l \end{bmatrix}$$

where $\Lambda_j = \begin{bmatrix} \alpha_j & \beta_j \\ -\beta_j & \alpha_j \end{bmatrix}$ with $\sigma_j = \alpha_j + i\beta_j$.

Solving $\mathbf{Y}' = \mathbf{B}\mathbf{Y}$ becomes

$$Y_1' = \lambda_1 Y_1$$

$$\vdots$$

$$Y_k' = \lambda_k Y_k$$

} exponential
decay/growth

$$Y_{k+1}' = \alpha_1 Y_{k+1} + \beta_1 Y_{k+2}$$

$$Y_{k+2}' = -\beta_1 Y_{k+1} + \alpha_2 Y_{k+2}$$

$$\vdots$$

$$Y_n' = -\beta_{n-1} Y_{n-1} + \alpha_n Y_n$$

} spiral growth/
decay or
rotation.

General solution:

$$Y_1(t) = C_1 e^{\lambda_1 t}$$

$$\vdots$$

$$Y_k(t) = C_k e^{\lambda_k t}$$

$$Y_{k+1}(t) = e^{\alpha_1 t} (C_{k+1} \cos \beta_1 t + C_{k+2} \sin \beta_1 t)$$

$$Y_{k+2}(t) = e^{\alpha_1 t} (C_{k+1} \sin(-\beta_1 t) + C_{k+2} \cos \beta_1 t)$$

$$\vdots$$

And $\Sigma(t; c) = P \Upsilon(t; c) \quad c \in \mathbb{R}^n.$

Case ③ If A is defective,
 then for each defective λ_j s.t.
 $a_j > g_j$ one can find $a_j - g_j$
 generalized EV for use on $P \in \mathbb{R}^{n \times n}$
 s.t.

$$B = P^{-1}AP = \begin{bmatrix} \Lambda_1 & & \\ & \ddots & \\ & & \Lambda_l \end{bmatrix}$$

where

$$\Lambda_j = \begin{bmatrix} \lambda_j & & \\ & \ddots & \\ & & \lambda_j \end{bmatrix} \in \mathbb{R}^{\nu \times \nu}, \quad \nu = a_j - g_j + 1$$

when $\lambda_j \in \mathbb{R}_j$

special case $v=1: \Lambda = [\lambda];$
 and for pairs of complex $\lambda \in \mathbb{C} \cup \overline{\lambda}:$

$$\Lambda = \begin{bmatrix} C & I_2 & & \\ & C & I_2 & \\ & & \ddots & \\ & & & I_2 \\ & & & & C \end{bmatrix} \in \mathbb{R}^{2v \times 2v}$$

where $v = \text{AM}(\lambda) - \text{GM}(\lambda) + 1$

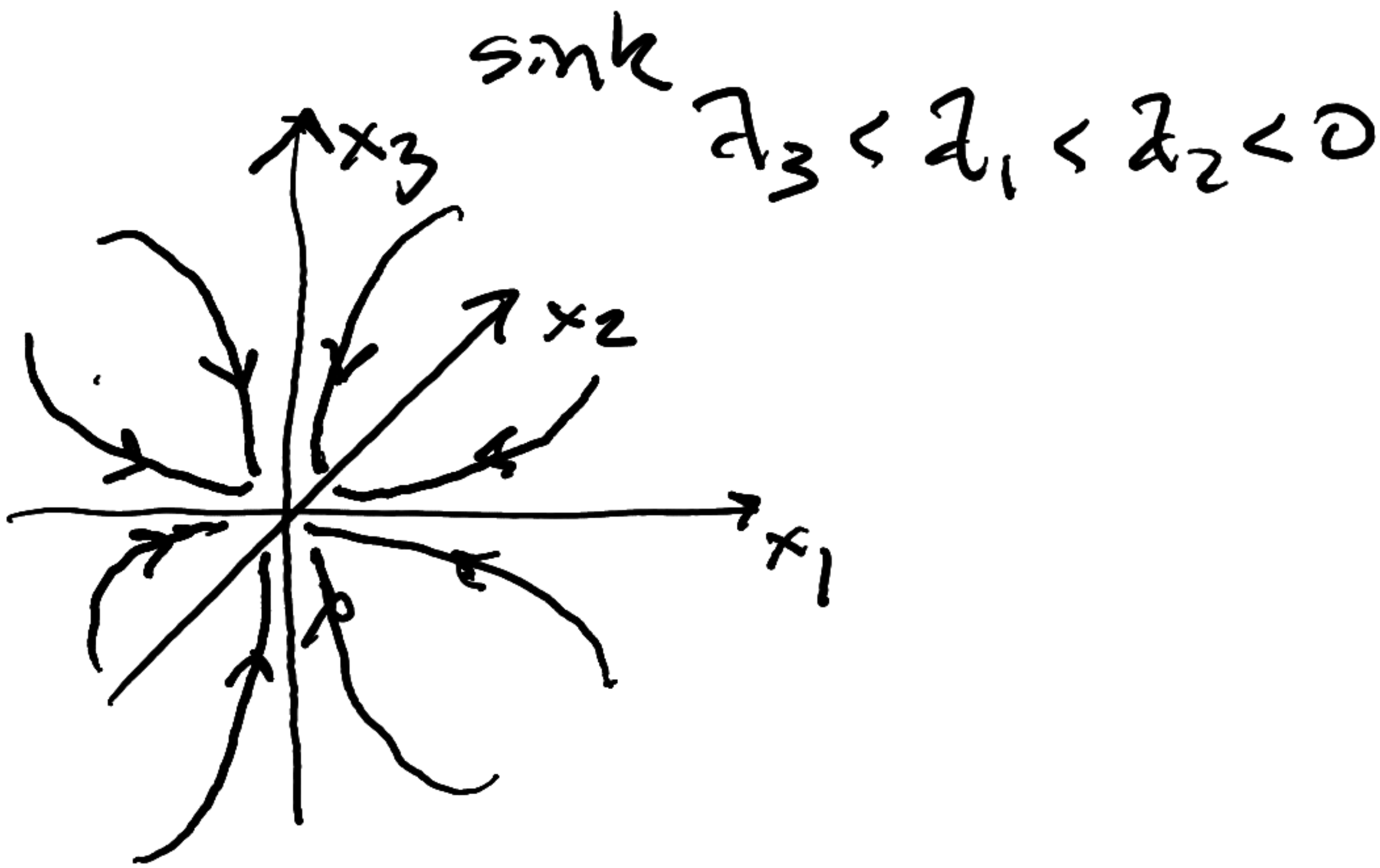
special case $v=1: \Lambda = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}.$

Ex 4 (Classification of EP)

If $A \in \mathbb{R}^{3 \times 3}$ diagonalizable and similar to $B = \begin{bmatrix} \lambda_1 & & \\ & \lambda_2 & \\ & & \lambda_3 \end{bmatrix}$, then

$$\mathbf{x}(t) = c_1 e^{\lambda_1 t} \mathbf{v}_1 + c_2 e^{\lambda_2 t} \mathbf{v}_2 + c_3 e^{\lambda_3 t}$$

If all $\lambda_i > 0$, then O is a source.
If all $\lambda_i < 0$, then O is a sink.



Ex 5

If $A \in \mathbb{R}^{3 \times 3}$ is defective with
one eigenpair (λ, v_1) s.t.

$AM(\lambda) = 3$ & $GM(\lambda) = 1$, then

solving

$$(A - \lambda I) V_2 = V_1 \quad \&$$

$$(A - \lambda I) V_3 = V_2 \quad \&$$

yields $P = [V_1 \ V_2 \ V_3]$ s.t.

$$B = P^{-1}AP = \begin{bmatrix} \lambda & 1 & 0 \\ 0 & \lambda & 1 \\ 0 & 0 & \lambda \end{bmatrix}$$

and $Y' = BY$ becomes

$$\left. \begin{aligned} Y_1' &= \lambda Y_1 + Y_2 \\ Y_2' &= \lambda Y_2 + Y_3 \\ Y_3' &= \lambda Y_3 \end{aligned} \right\} \Rightarrow \begin{aligned} Y_3 &= C_3 e^{\lambda t} \\ Y_2' - \lambda Y_2 &= C_3 e^{\lambda t} / e^{-\lambda t} \end{aligned}$$

$$\left(\begin{aligned} (Y_2 e^{-\lambda t})' &= C_3 \\ Y_2(t) &= (C_2 + C_3 t) e^{\lambda t} \\ Y_1' - \lambda Y_1 &= (C_2 + C_3 t) e^{\lambda t} / e^{-\lambda t} \\ (Y_1 e^{-\lambda t})' &= C_2 + C_3 t \\ Y_1(t) &= \left(C_1 + C_2 t + C_3 \frac{t^2}{2} \right) e^{\lambda t} \end{aligned} \right)$$

$$Y(t) = e^{\lambda t} \begin{bmatrix} C_1 + C_2 t + C_3 \frac{t^2}{2} \\ C_2 + C_3 t \\ C_3 \end{bmatrix}$$

General sol

$$X(t) = PZ(t).$$