

Lecture 12, 25/2

Matrix exponentials

Recall that $x' = ax$ $a \in \mathbb{R}$,
has general solution $x(t) = ce^{at}$, $c \in \mathbb{R}$.

Goal today: Explain extension that

$\underline{x}' = A\underline{x}$, $A \in \mathbb{R}^{n \times n}$ has general
solution $\underline{x}(t) = e^{At} C$, $C \in \mathbb{R}^n$.

Def For $A \in \mathbb{R}^{n \times n}$,
$$e^A := \sum_{j=0}^{\infty} \frac{A^j}{j!}, \text{ where } A^0 := I \in \mathbb{R}^{n \times n}.$$

Properties

- For any $j \geq 0$, $A^j \in \mathbb{R}^{n \times n}$.
- For any $N \geq 0$ $\sum_{j=0}^N \frac{A^j}{j!} \in \mathbb{R}^{n \times n}$.

• Theorem 1 will show that

$$e^A = \lim_{N \rightarrow \infty} \sum_{j=0}^N \frac{A^j}{j!} \in \mathbb{R}^{n \times n} \text{ holds for any } A \in \mathbb{R}^{n \times n}.$$

Example 1

(A^j not easy to compute in general, but easy for diagonal matrices.)

$$A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \Rightarrow A^j = \begin{bmatrix} \lambda_1^j & 0 \\ 0 & \lambda_2^j \end{bmatrix} \quad j \geq 0,$$

$$e^A = \lim_{N \rightarrow \infty} \sum_{j=0}^N \frac{A^j}{j!} = \lim_{N \rightarrow \infty} \begin{bmatrix} \sum_{j=0}^N \frac{\lambda_1^j}{j!} & 0 \\ 0 & \sum_{j=0}^N \frac{\lambda_2^j}{j!} \end{bmatrix}$$

$$\stackrel{(*)}{=} \begin{bmatrix} \sum_{j=0}^{\infty} \frac{\lambda_1^j}{j!} & 0 \\ 0 & \sum_{j=0}^{\infty} \frac{\lambda_2^j}{j!} \end{bmatrix} = \begin{bmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{bmatrix}.$$

So for $A = I$, $\lambda_1 = \lambda_2 = 1$ and

$$e^I = \begin{bmatrix} e & 0 \\ 0 & e \end{bmatrix}.$$

To justify convergence in (*) we need a norm on $\mathbb{R}^{n \times n}$.

Def The matrix 2-norm on $\mathbb{R}^{n \times n}$ is defined by

$$\|A\|_2 = \sup_{\underline{x} \in \mathbb{R}^n, \underline{x} \neq 0} \frac{|A\underline{x}|}{|\underline{x}|} = \max_{|\underline{x}|=1} |A\underline{x}|$$

$$\text{where } |\underline{x}| = \sqrt{\sum_{i=1}^n x_i^2}.$$

Ex If $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$, for any

unit vector $\underline{x}_\theta = (\cos \theta) \underline{e}_1 + (\sin \theta) \underline{e}_2$,

$$A\underline{x}_\theta = \begin{bmatrix} \cos \theta \lambda_1 \\ \sin \theta \lambda_2 \end{bmatrix} \quad \text{and}$$

$$\|A\|_2 = \max_{\theta \in [0, 2\pi)} |A\underline{x}_\theta| = \max(|\lambda_1|, |\lambda_2|).$$

In relation to (*)

$$\left\| \begin{bmatrix} \sum_{j=0}^N \frac{\lambda_1^j}{j!} & 0 \\ 0 & \sum_{j=0}^N \frac{\lambda_2^j}{j!} \end{bmatrix} - \begin{bmatrix} e^{\lambda_1} & 0 \\ 0 & e^{\lambda_2} \end{bmatrix} \right\|_2$$

$$\leq \max \left(\left| e^{\lambda_1} - \sum_{j=0}^N \frac{\lambda_1^j}{j!} \right|, \left| e^{\lambda_2} - \sum_{j=0}^N \frac{\lambda_2^j}{j!} \right| \right)$$

$\rightarrow 0$ as $N \rightarrow \infty$ (for any $\lambda_1, \lambda_2 \in \mathbb{R}$).

Properties of matrix 2-norm

- a) $\|A\mathbf{x}\| \leq \|A\|_2 \|\mathbf{x}\|$
- b) $(\mathbb{R}^{n \times n}, \|\cdot\|_2)$ is a Banach space: a complete, normed vector space.
- c) All norms on $\mathbb{R}^{n \times n}$ are equivalent, and $\|A\|_2 < \infty$ for all $A \in \mathbb{R}^{n \times n}$

Lemma 1 For $A, B \in \mathbb{R}^{n \times n}$, it holds that

a) $\|AB\|_2 \leq \|A\|_2 \|B\|_2$

b) $\|A^T\|_2 = \|A\|_2$.

Theorem 1 Let $A \in \mathbb{R}^{n \times n}$ and

consider the sequence

$$Z_N := \sum_{j=0}^N \frac{A^j}{j!} \text{ in } \mathbb{R}^{n \times n} \text{ for } N=0,1,\dots$$

Then $(Z_N)_{N \geq 0}$ converges in $\mathbb{R}^{n \times n}$

Proof By iteration $\|A^j\|_2 \stackrel{\text{Lemma 2a)}}{\leq} \|A\|_2^j$

for any $j \geq 0$. For $M > N \geq 0$,

$$\|Z_M - Z_N\|_2 = \left\| \sum_{j=N+1}^M \frac{A^j}{j!} \right\|_2$$

$$\leq \sum_{j=N+1}^M \frac{1}{j!} \|A^j\|_2 \leq \sum_{j=N+1}^M \frac{1}{j!} \|A\|_2^j$$

Since $\|A\|_2 < \infty$, there exists for any $\varepsilon > 0$ an $\bar{N}$ st.

$$\sum_{j=N+1}^{\infty} \frac{\|A\|_2^j}{j!} = \left| e^{\|A\|_2} - \sum_{j=0}^N \frac{\|A\|_2^j}{j!} \right| \leq \varepsilon \quad \forall N \geq \bar{N}.$$

Hence $\|Z_M - Z_N\|_2 \leq \varepsilon \quad \forall M, N \geq \bar{N}$,

so (Z_N) is Cauchy in $(\mathbb{R}^{n \times n}, \|\cdot\|_2)$. ■

Ex $A = \begin{bmatrix} 0 & B \\ -B & 0 \end{bmatrix} = B \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_{=: \hat{I}}$

$A^j = B^j \hat{I}^j$ where $\hat{I}^2 = -I, \hat{I}^3 = -\hat{I},$

$\hat{I}^4 = I, \dots$, so that

$A^{2j} = (-1)^j B^{2j} I, A^{2j+1} = (-1)^j B^{2j+1} \hat{I}$ and

$$e^A = \sum_{j=0}^{\infty} \frac{A^{2j}}{(2j)!} + \sum_{j=0}^{\infty} \frac{A^{2j+1}}{(2j+1)!} =$$

$$= \underbrace{\sum_{j=0}^{\infty} \frac{(-1)^j \beta^{2j}}{(2j)!}}_{\cos \beta} I + \underbrace{\sum_{j=0}^{\infty} \frac{(-1)^j \beta^{2j+1}}{(2j+1)!}}_{\sin \beta} \hat{I}$$

$$= \begin{bmatrix} \cos \beta & \sin \beta \\ -\sin \beta & \cos \beta \end{bmatrix}.$$

It follows that for $A = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$

$$e^{tA} = e^{\begin{bmatrix} t\lambda_1 & 0 \\ 0 & t\lambda_2 \end{bmatrix}} = \begin{bmatrix} e^{t\lambda_1} & 0 \\ 0 & e^{t\lambda_2} \end{bmatrix}$$

$$\text{so that } \underline{x}(t) := e^{tA} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 e^{t\lambda_1} \\ c_2 e^{t\lambda_2} \end{bmatrix}$$

indeed is the general solution of

$$\underline{x}' = A\underline{x}.$$

And for $A = \begin{bmatrix} 0 & B \\ -B & 0 \end{bmatrix}$,

$$e^{At} = \begin{bmatrix} \cos Bt & \sin Bt \\ -\sin Bt & \cos Bt \end{bmatrix} \quad \text{so that}$$

$$X(t) = e^{At} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = c_1 \begin{bmatrix} \cos Bt \\ -\sin Bt \end{bmatrix} + c_2 \begin{bmatrix} \sin Bt \\ \cos Bt \end{bmatrix}$$

is the general solution of $X' = AX$.

Lemma 2 Let $A, B, P \in \mathbb{R}^{n \times n}$ with P invertible. Then

a) If $B = P^{-1}AP$, then $e^B = P^{-1}e^A P$

b) If $AB = BA$, then $e^{A+B} = e^A e^B$

c) $(e^A)^{-1} = e^{-A}$.

Note If $P^{-1}AP = \begin{bmatrix} \lambda_1 & & \\ & \ddots & \\ & & \lambda_n \end{bmatrix} = B$, then

a) $\Rightarrow e^A = P e^B P^{-1} = P \begin{bmatrix} e^{\lambda_1} & & \\ & \ddots & \\ & & e^{\lambda_n} \end{bmatrix} P^{-1}$.

Proof a) $e^B = \sum_{j=0}^{\infty} \frac{(P^{-1}AP)^j}{j!}.$

Claim: $(P^{-1}AP)^j = P^{-1}A^jP, \forall j \geq 0$

Holds for $j \leq 1$. If true for j , then

$$\begin{aligned} (P^{-1}AP)^{j+1} &= (P^{-1}AP)^j (P^{-1}AP) \\ &= P^{-1}A^j \underbrace{P P^{-1}}_I AP = P^{-1}A^{j+1}P. \end{aligned}$$

Holds by induction.

$$\begin{aligned} \Rightarrow e^B &= \lim_{N \rightarrow \infty} \sum_{j=0}^N \frac{(P^{-1}AP)^j}{j!} \\ &= \lim_{N \rightarrow \infty} P^{-1} \left(\sum_{j=0}^N \frac{A^j}{j!} \right) P \end{aligned}$$

$$= P^{-1} \left(\lim_{N \rightarrow \infty} \sum_{j=0}^N \frac{A^j}{j!} \right) P \quad \begin{matrix} \text{(Lemma 1a)} \\ \text{(Theorem 1)} \end{matrix}$$

$$= P^{-1}e^A P$$

b) Need to show that

$$\sum_{n=0}^{\infty} \frac{(A+B)^n}{n!} = e^{A+B} = e^A e^B = \left(\sum_{j=0}^{\infty} \frac{A^j}{j!} \right) \left(\sum_{k=0}^{\infty} \frac{B^k}{k!} \right)$$

Since $AB=BA$,

$$(A+B)^2 = A^2 + AB + BA + B^2 = A^2 + 2AB + B^2,$$

and

$$(A+B)^n = \sum_{j=0}^n \binom{n}{j} A^j B^{n-j} \quad j \geq 2$$

By Lemma 1a) and Theorem 1,

$$e^A e^B \stackrel{(**)}{=} \lim_{N \rightarrow \infty} \sum_{j=0}^N \sum_{k=0}^N \frac{1}{j! k!} A^j B^k$$

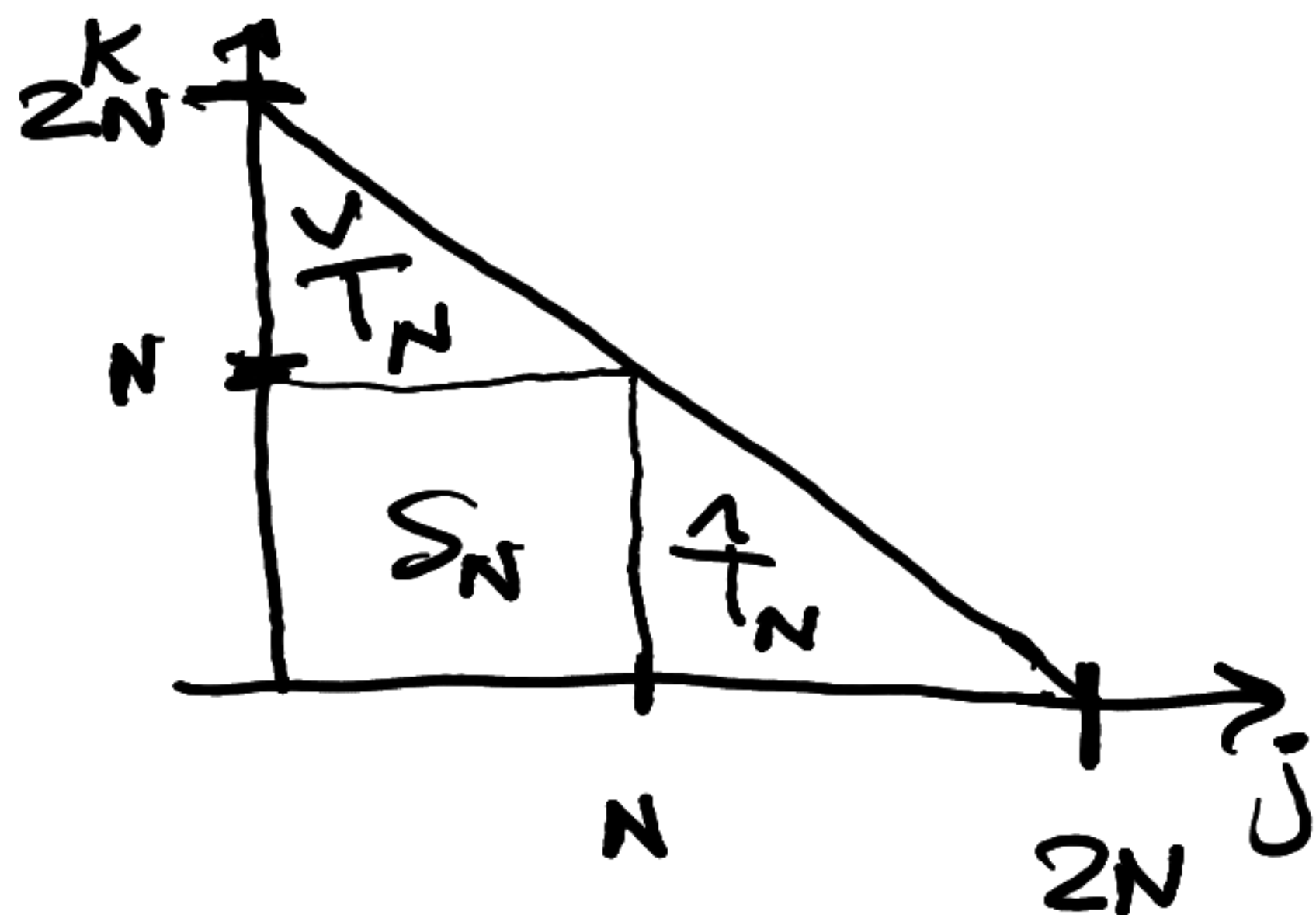
$$= \lim_{N \rightarrow \infty} \sum_{(j,k) \in S_N} \frac{1}{j! k!} A^j B^k$$

where $S_N = \{(j,k) \in \mathbb{N}_0^2 \mid j \leq N, k \leq N\}$.

Let $T_N = \{(j,k) \in \mathbb{N}_0^2 \mid j+k \leq 2N\}$, and

note that

$$T_N = S_N \cup \overset{\vee}{T}_N \cup \hat{T}_N$$



$$\hat{T}_N = \{(j,k) \mid j > N, j+k \leq 2N\} \subset \{N+1, \dots, 2N\} \times \{1, \dots, N\}$$

$$\overset{\vee}{T}_N = \{(j,k) \mid k > N, j+k \leq 2N\} \subset \{0, \dots, N\} \times \{N+1, \dots, 2N\}$$

$$\begin{aligned} \left\| \sum_{(j,k) \in \hat{T}_N} \frac{A^j B^k}{j! k!} \right\|_2 &\leq \sum_{(j,k) \in \hat{T}_N} \frac{\|A\|_2^j \|B\|_2^k}{j! k!} \\ &\leq \left(\sum_{j=N+1}^{2N} \frac{\|A\|_2^j}{j!} \right) \underbrace{\sum_{k=0}^N \frac{\|B\|_2^k}{k!}}_{\leq e^{\|B\|_2}} \\ &\xrightarrow{N \rightarrow \infty} 0 \end{aligned}$$

and similarly

$$\left\| \sum_{(j,k) \in \overset{\vee}{T}_N} \frac{A^j B^k}{j! k!} \right\|_2 \rightarrow 0 \quad \text{as } N \rightarrow \infty.$$

Hence

$$e^A e^B = \lim_{N \rightarrow \infty} \sum_{(j,k) \in S_N} \frac{A^j B^k}{j! k!} = \lim_{N \rightarrow \infty} \sum_{(j,k) \in T_N} \frac{A^j B^k}{j! k!}$$

$$= \lim_{N \rightarrow \infty} \sum_{n=0}^{2N} \sum_{j+k=n} \frac{A^j B^k}{j! k!} \quad \boxed{k=n-j}$$

$$= \lim_{N \rightarrow \infty} \sum_{n=0}^{2N} \frac{1}{n!} \sum_{j=0}^n \frac{n!}{j!(n-j)!} A^j B^{n-j}$$

$$= \lim_{N \rightarrow \infty} \sum_{n=0}^{2N} \frac{1}{n!} (A+B)^n = e^{A+B}.$$

c) Note that $O \in \mathbb{R}^{n \times n}$ satisfies

$$O^j = \begin{cases} I & j=0 \\ 0 & j \geq 1 \end{cases}$$

$$\text{so } e^O = \sum_{j=0}^{\infty} \frac{O^j}{j!} = I.$$

Since A and $(-A)$ commute,

$$e^A e^{-A} \stackrel{b)}{=} e^{A-A} = I.$$



(**) holds as for

$$\left. \begin{aligned} Y_N &= \sum_{j=0}^N \frac{A^j}{j!} \\ Z_N &= \sum_{k=0}^N \frac{B^k}{k!} \end{aligned} \right\} \text{Thm 1 } Y_N \rightarrow e^A \\ \Rightarrow Z_N \rightarrow e^B$$

And any $N \in \mathbb{N}$ (N being finite),

$$Y_N Z_N = \sum_{j=0}^N \sum_{k=0}^N \frac{A^j B^k}{j! k!}$$

(order of sum does not matter),

and

$$\|e^A e^B - Y_N Z_N\|_2 \leq \|e^A (e^B - Z_N)\|_2 \\ + \|(e^A - Y_N) Z_N\|_2$$

Lemma

$$\leq \underbrace{\|e^A\|_2}_{\leq \infty} \underbrace{\|e^B - Z_N\|_2}_{\rightarrow 0} + \underbrace{\|e^A - Y_N\|_2}_{\rightarrow 0} \underbrace{\|Z_N\|_2}_{\leq e^{\|B\|_1}} \rightarrow 0$$

Exercises

6.1 Find general solution of $\underline{X}' = A\underline{X}$ for

$$a) A = \begin{bmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad b) \begin{bmatrix} 1 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}$$

$$a) \begin{array}{l} x_1' = x_3 \\ x_2' = x_2 \\ x_3' = x_1 \end{array} \quad \left| \quad \det(A - \lambda I) = \det \begin{bmatrix} -\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & -\lambda \end{bmatrix} \right.$$

$$= -\lambda \begin{vmatrix} 1-\lambda & 0 \\ 0 & -\lambda \end{vmatrix} + 1 \cdot \begin{vmatrix} 0 & 1-\lambda \\ 1 & 0 \end{vmatrix}$$

$$= \lambda^2(1-\lambda) + \lambda - 1 = (\lambda - 1)(1 - \lambda^2)$$

$$= -(\lambda - 1)^2(\lambda + 1)$$

$$EV \quad \lambda_1 = -1, \lambda_2 = 1, \lambda_3 = 1$$

Linearly indep eigenvectors:

$$\begin{bmatrix} 1 & 0 & 1 \\ 0 & 2 & 0 \\ 1 & 0 & 1 \end{bmatrix} V_1 = 0, \quad V_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$$

$$\text{and } V_2 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}, \quad V_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}.$$

$$X(t) = c_1 e^{-t} V_1 + c_2 e^t V_2 + c_3 e^t V_3$$

$$= \begin{bmatrix} c_1 e^{-t} + c_2 e^t \\ c_3 e^t \\ -c_1 e^{-t} + c_2 e^t \end{bmatrix} \quad c \in \mathbb{R}^3$$

$$b) \det(A - \lambda I) = \det \begin{bmatrix} 1-\lambda & 0 & 1 \\ 0 & 1-\lambda & 0 \\ 1 & 0 & 1-\lambda \end{bmatrix}$$

$$= (1-\lambda) \begin{vmatrix} 1-\lambda & 0 \\ 0 & 1-\lambda \end{vmatrix} + 1 \cdot \begin{vmatrix} 0 & 1-\lambda \\ 1 & 0 \end{vmatrix}$$

$$= (1-\lambda)^3 + \lambda - 1$$

$$= (1-\lambda) \left((1-\lambda)^2 - 1 \right)$$

$$= (1-\lambda)(-\lambda)(2-\lambda) = 0$$

$$\text{EV: } \lambda_1 = 0, \lambda_2 = 1, \lambda_3 = 2$$

$$V_1 = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \quad V_2 = \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix}, \quad V_3 = \begin{bmatrix} 1 \\ 0 \\ 1 \end{bmatrix}$$

$$\vec{x}(t) = c_1 V_1 + c_2 e^t V_2 + c_3 e^{2t} V_3$$

$$= \begin{bmatrix} c_1 + c_3 e^{2t} \\ c_2 e^t \\ -c_1 + c_3 e^{2t} \end{bmatrix}$$

6.2 $\underline{X}' = \begin{bmatrix} \lambda_1 & 0 & 0 \\ 0 & \lambda_2 & 0 \\ 0 & 0 & \lambda_3 \end{bmatrix}$ with $\lambda_3 < \lambda_2 < \lambda_1 < 0$.

Describe how $\underline{X}(t; \underline{X}_0)$ approaches 0 as $t \rightarrow \infty$ for arbitrary $\underline{X}_0 = \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix}$.

Solution $\underline{X}(t) = \begin{bmatrix} x_1 e^{\lambda_1 t} \\ x_2 e^{\lambda_2 t} \\ x_3 e^{\lambda_3 t} \end{bmatrix}$

If $x_1 \neq 0$, then

$$\frac{|x_2|e^{\lambda_2 t} + |x_3|e^{\lambda_3 t}}{|x_1|e^{\lambda_1 t}} \leq \frac{|x_2| + |x_3|}{|x_1|} e^{(\lambda_2 - \lambda_1)t} \rightarrow 0 \text{ as } t \rightarrow \infty$$

$\Rightarrow 0$ approached along E_1 -direction

If $x_1 = 0$, but $x_2 \neq 0$, then

$$\frac{|x_3|e^{\lambda_3 t}}{|x_2|e^{\lambda_2 t}} \rightarrow 0 \text{ as } t \rightarrow \infty$$

so 0 approached in E_2 -direction.

If $x_1 = x_2 = 0$, then 0 approached in E_3 -dir.

6.3 Give an example of an $A \in \mathbb{R}^{3 \times 3}$ for which all non-equilibrium solutions of $\underline{x}' = A\underline{x}$ are 2π -periodic. Sketch phase portrait.

Sol The planar system $\hat{\underline{x}}' = \begin{bmatrix} 0 & \beta \\ \beta & 0 \end{bmatrix} \hat{\underline{x}}$, $\beta > 0$, has general solution

$$\hat{\underline{x}}(t) = C_1 \begin{bmatrix} \cos \beta t \\ -\sin \beta t \end{bmatrix} + C_2 \begin{bmatrix} \sin \beta t \\ \cos \beta t \end{bmatrix}.$$

$$\hat{\underline{x}}\left(t + \frac{2\pi}{\beta}\right) = \hat{\underline{x}}(t), \text{ so } \hat{\underline{x}} \text{ is } \frac{2\pi}{\beta} \text{ periodic}$$

Let us embed this dynamics in $\mathbb{R}^3$ with

$\beta = 1$. Consider

$$\underline{X}' = \begin{bmatrix} 0 & 1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} \underline{X} = \begin{bmatrix} \begin{bmatrix} 0 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \\ 0 \end{bmatrix}$$

Two uncoupled problems

$$\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \Rightarrow \begin{aligned} x_1 &= c_1 \cos t + c_2 \sin t \\ x_2 &= -c_1 \sin t + c_2 \cos t \end{aligned}$$

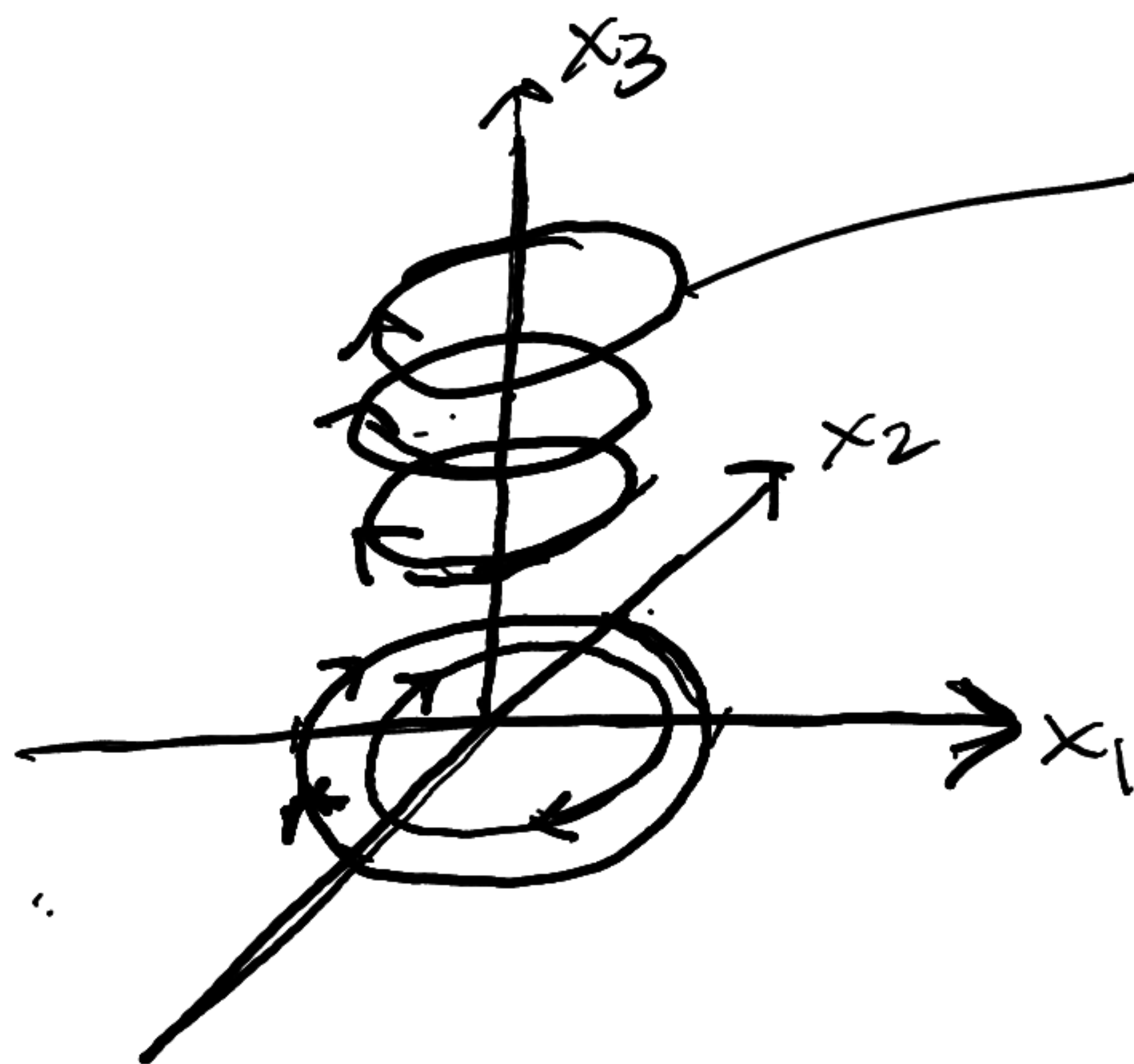
$$x_3' = 0 \Rightarrow x_3 = c_3$$

$\underline{X}$ is a non-equilibrium sol iff

$$A \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix} \neq 0 \Leftrightarrow \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \neq \begin{bmatrix} 0 \\ 0 \end{bmatrix}.$$

All non-eq solutions are 2π -periodic

Phase portrait



similar centers
in x_1x_2 -plane at
different x_3 -values

$$A^2 = \begin{bmatrix} \hat{I}^2 & \hat{I} \\ 0 & \hat{I}^2 \end{bmatrix},$$

$$A^3 = \begin{bmatrix} \hat{I}^3 & 2\hat{I}^2 \\ 0 & \hat{I}^3 \end{bmatrix}$$