

Lecture 13 MAT 3440

Recall For $A \in \mathbb{R}^{n \times n}$, $e^A \in \mathbb{R}^{n \times n}$ is defined by

$$e^A = \sum_{j=0}^{\infty} \frac{A^j}{j!}$$

and

Lemma 1 Let $A, B, P \in \mathbb{R}^{n \times n}$

a) If $B = P^{-1}AP$, then $e^B = P^{-1}e^A P$

b) If $AB = BA$, then $e^{A+B} = e^A e^B$

c) $(e^A)^{-1} = e^{-A}$

Note for any $A \in \mathbb{R}^{n \times n}$, e^A is invertible.

Lemma 2 If (λ, V) is an eigenpair of $A \in \mathbb{R}^{n \times n}$, then (e^λ, V) is an eigenpair of e^A . (May have $\lambda \in \mathbb{C}$, $V \in \mathbb{C}^n$.)

Proof For $Z_N = \sum_{j=0}^N \frac{A^j}{j!}$, it holds that

$$\|Z_N - e^A\|_2 \rightarrow 0 \text{ as } N \rightarrow \infty.$$

$$\Rightarrow \|Z_N V - e^A V\| \leq \|Z_N - e^A\|_2 \|V\| \xrightarrow{N \rightarrow \infty} 0$$

where $Z_N V = \sum_{j=0}^N \frac{A^j V}{j!}$. Hence,

$$e^A V = \left(\lim_{N \rightarrow \infty} \sum_{j=0}^N \frac{A^j}{j!} \right) V = \lim_{N \rightarrow \infty} \underbrace{\left(\sum_{j=0}^N \frac{A^j V}{j!} \right)}_{Z_N V}$$

and

$$e^A V = \left(\lim_{N \rightarrow \infty} \sum_{j=0}^N \frac{A^j}{j!} \right) V$$

$$= \lim_{N \rightarrow \infty} \left(\sum_{j=0}^N \frac{A^j V}{j!} \right) = \lim_{N \rightarrow \infty} \left(\sum_{j=0}^N \frac{A^j V}{j!} \right)$$

$$= e^A V.$$



Consider $\dot{X} = AX$, $A \in \mathbb{R}^{n \times n}$.

We have shown (Lecture 5) that

$X(t) = e^{At}V$ is a solution,

and Lemma 2 $\Rightarrow e^{2t}V = e^{At}V$.

Will show that $X(t) = e^{At}C$ for any $C \in \mathbb{R}^n$ is a solution.

Lemma 3 Consider $F(t) = e^{tA}$, so $F: \mathbb{R} \rightarrow \mathbb{R}^{n \times n}$.

Then $F'(t) = \frac{d}{dt} e^{tA} = A e^{tA} = e^{tA} A$.

Proof For any $h \in \mathbb{R}$, $(t+h)A = tA + hA$
where tA and hA commute. Lemma 1 b)

$$\Rightarrow e^{(t+h)A} = e^{tA} e^{hA}.$$

$$\frac{d}{dt} e^{At} = \lim_{h \rightarrow 0} \frac{e^{(t+h)A} - e^{tA}}{h}$$

$$= \lim_{h \rightarrow 0} \frac{e^{tA} (e^{hA} - I)}{h}$$

$$= e^{tA} \left(\lim_{h \rightarrow 0} \frac{e^{hA} - I}{h} \right) = e^{tA} A$$

as
$$e^{hA} - I = \sum_{j=1}^{\infty} (hA)^j = hA + h^2 \sum_{j=2}^{\infty} A^j h^{j-2}$$

$$= hA + O(h^2).$$

Moreover

$$e^{tA} A = \left(\lim_{N \rightarrow \infty} \sum_{j=0}^N \frac{t^j A^j}{j!} \right) A$$

$$= \lim_{N \rightarrow \infty} \left(\sum_{j=0}^N \frac{t^j A^{j+1}}{j!} \right)$$

$$= A \left(\lim_{N \rightarrow \infty} \sum_{j=0}^N \frac{t^j A^j}{j!} \right) = A e^{tA}.$$

Theorem 1 $\underline{X}' = A\underline{X}$ with $A \in \mathbb{R}^{n \times n}$ has
general solution $\underline{X}(t) = e^{At} C$ $C \in \mathbb{R}^n$.
And $\underline{X}(t) = e^{At} \underline{X}_0$ is the unique
solution in $C^1(\mathbb{R}, \mathbb{R}^n)$ satisfying
 $\underline{X}(0) = \underline{X}_0$.

Proof $e^{At} C$ is a C^1 -solution:

$$\underline{X}'(t) = \left(\frac{d}{dt} e^{At} \right) C$$

$$\stackrel{\text{Lemma 3}}{=} A e^{At} C = A \underline{X}(t).$$

Suppose $\underline{Y}(t)$ is another

Then $\underline{Z}(t) = e^{-tA} \underline{Y}(t)$ satisfies

$$\begin{aligned} \underline{Z}'(t) &= \left(\frac{d}{dt} e^{-tA} \right) \underline{Y} + e^{-tA} \underline{Y}' \\ &= e^{-tA} (-A) \underline{Y} + e^{-tA} A \underline{Y} = 0. \end{aligned}$$

Hence $Z(t) = C$ for some $C \in \mathbb{R}^n$,
 and $e^{At} C = e^{At} Z = \underbrace{e^{At} e^{-At}}_{\substack{\text{Lemma 1} \\ \underline{\underline{I}}}} Y(t)$

$\Rightarrow$ any solution of the ODE is on the form

$$Y(t) = e^{At} C.$$

Lastly, the initial condition $X(0) = X_0$

$$\Rightarrow \underbrace{e^{A \cdot 0}}_I C = X_0, \text{ which determine}$$

the degrees of freedom, C , uniquely.



Ex 1 $X' = \underbrace{\begin{bmatrix} \lambda & 1 \\ 0 & \lambda \end{bmatrix}}_A X.$

Observe $A^2 = \begin{bmatrix} \lambda^2 & 2\lambda \\ 0 & \lambda^2 \end{bmatrix}, A^j = \begin{bmatrix} \lambda^j & j\lambda^{j-1} \\ 0 & \lambda^j \end{bmatrix} \quad j \geq 1.$

$$\text{So } (tA)^j = t^j A^j = \begin{bmatrix} (t\lambda)^j & t^j (t\lambda)^{j-1} \\ 0 & (t\lambda)^j \end{bmatrix} \quad j \geq 1$$

$$e^{tA} = \sum_{j=0}^{\infty} \frac{(tA)^j}{j!}$$

$$\stackrel{(*)}{=} \begin{bmatrix} \sum_{j=0}^{\infty} \frac{(t\lambda)^j}{j!} & t \sum_{j=1}^{\infty} \frac{(t\lambda)^{j-1}}{(j-1)!} \\ 0 & \sum_{j=0}^{\infty} \frac{(t\lambda)^j}{j!} \end{bmatrix}$$

$$= \begin{bmatrix} e^{\lambda t} & t e^{\lambda t} \\ 0 & e^{\lambda t} \end{bmatrix}$$

$$\text{So } \underline{X}(t) = e^{At} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} c_1 e^{\lambda t} + c_2 t e^{\lambda t} \\ c_2 e^{\lambda t} \end{bmatrix},$$

as we already knew.

Nonautonomous and/or inhomogeneous

$$\bar{X}' = A(t)\bar{X} + G(t) = (F(\bar{X}, t)),$$

where $G \in C(\mathbb{R}, \mathbb{R}^n)$ and
 $A \in C(\mathbb{R}, \mathbb{R}^{n \times n})$.

Will here restrict to $A \in \mathbb{R}^{n \times n}$
time independent:

$$\bar{X}' = A\bar{X} + G(t) \quad (1)$$

Lemma 4

If Z solves (1), then

$$\bar{X} = Z + e^{At}C \quad C \in \mathbb{R}^n$$

is the general solution of (1).

Proof Z is a particular solution of
(1) and $\bar{Y} = e^{At} C$ is a
solution of the homogeneous
part $\bar{Y}' = A\bar{Y}$. For $X = \bar{Y} + Z$, we

$$\text{obtain } X' = \bar{Y}' + Z' = A\bar{Y} + AZ + G \\ = AX + G.$$

Suppose $\hat{X}$ is another solution: $\hat{X}' = A\hat{X} + G$.

Then $W = X - \hat{X}$ satisfies

$$W' = X' - \hat{X}' = A(X - \hat{X}) + (G - G) \\ = AW.$$

Thm 1
 $\Rightarrow W = e^{At} \hat{C}$ for some $\hat{C} \in \mathbb{R}^n$

$$\text{so } \hat{X} = X + W = \bar{Y} + e^{At} \hat{C} + Z \\ = e^{At} (C + \hat{C}) + Z.$$

□

Theorem 2

$$\underline{Y(t) = e^{tA} \left(X_0 + \int_0^t e^{-sA} G(s) ds \right)}$$

is a particular C^1 -solution of (1)

and $Y(0) = X_0$.

Note $\int_0^t e^{-sA} G(s) ds = \begin{bmatrix} \int_0^t (e^{-sA} G(s))_1 ds \\ \vdots \\ \int_0^t (e^{-sA} G(s))_n ds \end{bmatrix}$

Proof

$$\begin{aligned} Y'(t) &= \left(\frac{d}{dt} e^{tA} \right) \left(X_0 + \int_0^t e^{-sA} G(s) ds \right) \\ &\quad + e^{tA} \left(\frac{d}{dt} \int_0^t e^{-sA} G(s) ds \right) \\ &= A e^{tA} \left(X_0 + \int_0^t e^{-sA} G(s) ds \right) + \underbrace{e^{tA} e^{-tA}}_{=I} G(t) \end{aligned}$$

$$= A\mathbb{Y} + G.$$

$$\text{And } \mathbb{Y}(0) = e^0 (\mathbb{X}_0 + 0) = \mathbb{X}_0.$$

~~□~~

Ex 2 Harmonic oscillator
with periodic force

$$x'' + x = \cos t, \quad x(0) = 0, \quad x'(0) = 1$$

$$\begin{aligned} x_1 &= x \\ x_2 &= x' \end{aligned} \Rightarrow \mathbb{X}' = \underbrace{\begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}}_A \mathbb{X} + \underbrace{\begin{bmatrix} 0 \\ \cos t \end{bmatrix}}_G$$

$$\text{and } \mathbb{X}(0) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}.$$

Recall that

$$e^{At} = e^{\begin{bmatrix} 0 & t \\ -t & 0 \end{bmatrix}} = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$$

Solution:

$$X(t) = e^{At} \left(\begin{bmatrix} 0 \\ 1 \end{bmatrix} + \int_0^t e^{-As} G(s) ds \right)$$

$$= \begin{bmatrix} \sin t \\ \cos t \end{bmatrix} + \underbrace{\int_0^t e^{A(t-s)} G(s) ds}_{I(t)}$$

$$\text{where } I(t) = \begin{bmatrix} \int_0^t \sin(t-s) \cos(s) ds \\ \int_0^t \cos(t-s) \cos(s) ds \end{bmatrix}$$

$$= \begin{bmatrix} \frac{1}{2} \int_0^t \sin t + \sin(t-2s) ds \\ \frac{1}{2} \int_0^t \cos t + \cos(t-2s) ds \end{bmatrix}$$

$$= \frac{1}{2} \begin{bmatrix} t \sin t \\ t \cos t + \sin t \end{bmatrix}$$

and $\underline{x}(t) = \begin{bmatrix} \sin t + t \frac{\sin t}{2} \\ \cos t + t \frac{\cos t}{2} + \frac{\sin t}{2} \end{bmatrix}$

with $x(t) = x_1(t)$.

Note

- For $x'' = -Kx$, $\sqrt{K} > 0$ is the natural/eigen frequency of the oscillator.
- When $K=1$, a periodic force with the same frequency $\cos(\omega t)$ with $\omega = 1 = \sqrt{K}$ leads to resonance and $\sup_{t \in \mathbb{R}} |x(t)| = \infty$.
- If $\omega \neq 1$, then $\sup_{t \in \mathbb{R}} |x(t)| < \infty$,
see § 6.5
- And for $x'' + bx' + x = \cos t$ with $b > 0$ leads to $x(t) = \gamma(t) + \hat{x}(t)$ where $|\gamma(t)| \rightarrow 0$ as $t \rightarrow \infty$ and $\hat{x}$ is 2π -periodic. See § 6.5.