

Lecture 14, MAT 3440 9/3

Not all questions can be answered "well" in different ways:

1. Let $m+n=10$ for $m, n \in \mathbb{N}$,
what is m and n ?
(no uniqueness.)

2. Find $x, y \in \mathbb{R}$ s.t. $x^2 + y^2 = -1$,
(no existence)

3. Too little information/instability:
What will the exact temperature
be in Oslo this day in year 2051?

Def (Hadamard 1902)

The IVP $\underline{X}' = F(t, \underline{X}) \quad t \in I$
 $\underline{X}(t_0) = \underline{X}_0 \in \mathbb{R}^n$

is called well-posed if

1. A solution $\underline{X} \in C^1(I, \mathbb{R}^n)$ exists.

2. The solution is unique.

3. The solution depends continuously on $\underline{X}_0$, meaning

$$\underline{X}(t; \underline{X}_0) - \underline{X}(t; \tilde{\underline{X}}_0) \rightarrow 0 \text{ as}$$

$$\tilde{\underline{X}}_0 \rightarrow \underline{X}_0.$$

Otherwise, the IVP is called ill-posed.

Point 3. is important if $\mathbb{X}_0$ comes from measurements or if one solves the ODE numerically, as one often has

$$\underbrace{\bar{X}(t; \mathbb{X}_0)}_{\text{num sol}} = \underbrace{X(t; \tilde{\mathbb{X}}_0)}_{\text{exact sol}}, \quad \tilde{\mathbb{X}}_0 = \mathbb{X}_0 + O(\Delta t).$$

Plan today

Examples of IVPs failing 1. & 2.
and Picard iteration.

Wed Picard-Lindelöf Thm: local
existence & uniqueness (for
autonomous ODE).

Recall For intervals $I \subset \mathbb{R}$,

$$C'(I, \mathbb{R}^n) := \{X: I \rightarrow \mathbb{R}^n \mid x_1, \dots, x_n \in C'(I)\}$$

and for a domain (open and connected) $D \subset \mathbb{R}^n$

$$C'(D, \mathbb{R}^n) := \{F: D \rightarrow \mathbb{R}^n \mid f_i: D \rightarrow \mathbb{R} \text{ satisfies } f_i, \partial_{x_j} f_i \in C(D) \text{ for all } i, j \in \{1, \dots, n\}\}$$

Theorem 1 (Local existence & uniqueness)

Consider the IVP

$$\left. \begin{array}{l} X' = F(X) \\ X(t_0) = X_0 \in \mathbb{R}^n \end{array} \right\} (1)$$

with $F: \mathbb{R}^n \rightarrow \mathbb{R}^n$. If $F \in C'(\mathbb{R}^n, \mathbb{R}^n)$,

then there exists an $I = [t_0 - a, t_0 + a]$ for some $a > 0$ and a unique solution $X \in C'(I, \mathbb{R}^n)$ to (1) (on I).

Examples in 1D

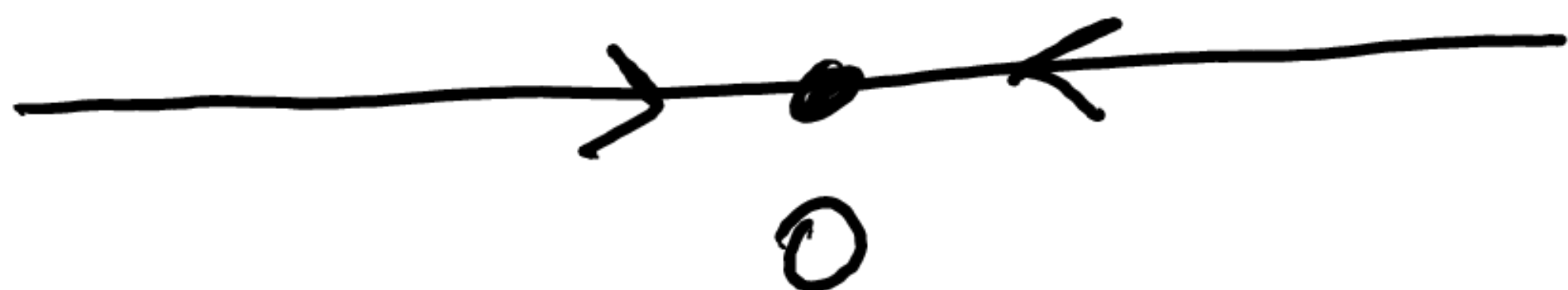
Ex 1 (No existence)

$$x' = f(x) = \begin{cases} 1 & x < 0 \\ -1 & x \geq 0 \end{cases} = -\operatorname{sign}(x)$$

$$x(0) = 0$$

Here $f \notin C(\mathbb{R})$, so Thm 1 does not apply,

0 is "almost" an EP, but $f(0) \neq 0$.



Suppose a solution $x \in C^1([-a, a])$ for some $a > 0$. Then

$$\begin{aligned} \frac{d}{dt} (x(t))^2 &= 2x' \cdot x = -2 \operatorname{sign}(x) x \\ &= -2|x| \leq 0 \end{aligned}$$

$$\Rightarrow x(t)^2 \leq x^2(0) = 0 \Rightarrow x(t) = 0, t \geq 0$$

But then $x'(t) = 0 \neq f(x(t)) = -1 \quad t \geq 0$,

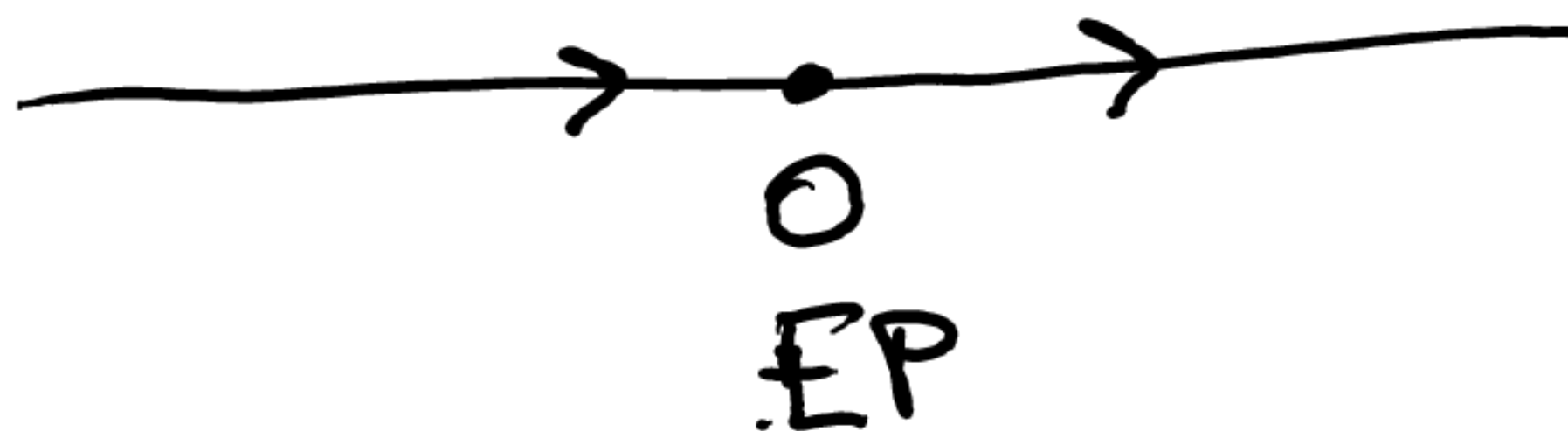
Contradiction (does not solve ODE).

Ex2 No uniqueness

$$x' = \sqrt{|x|} =: f(x)$$

$$x(0) = 0$$

Phase line

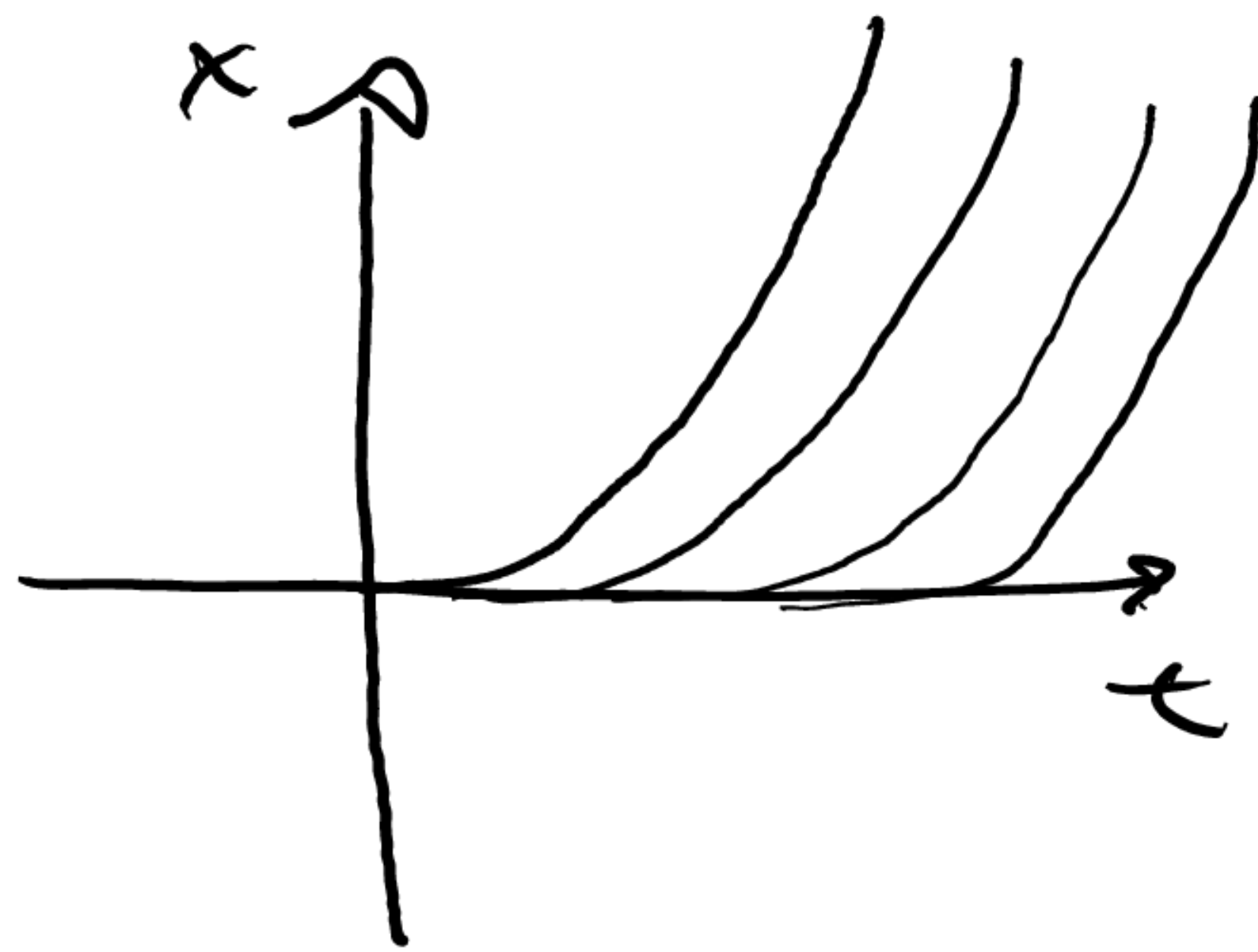


$x(t) = 0$ is a $C^1(\mathbb{R})$ solution, as

$$x'(t) = 0 = \sqrt{|x(t)|}.$$

But for any $\tau > 0$,

$$x(t) = \begin{cases} 0 & t \leq \tau \\ \frac{(t-\tau)^2}{4} & t > \tau \end{cases}$$



is also a C^1 solution, as

$$x'(\tau-) = 0 = x'(\tau+), \text{ and for } t > \tau$$

$$x'(t) = \frac{t-\tau}{2} = \sqrt{|x(t)|}. \quad (\text{Q: any more solutions?})$$

(Again $f \notin C^1(\mathbb{R})$, so Thm 1 does not apply.)

Picard iteration

Given IVP

$$\underline{X}' = F(\underline{X})$$

$$\underline{X}(0) = \underline{X}_0$$

Let $U_k : [-a, a] \rightarrow \mathbb{R}^n$ for some $a > 0$,
be defined by

$$U_0(t) = \underline{X}_0 \quad \forall t \quad \text{and}$$

$$U_{k+1}(t) = \underline{X}_0 + \int_0^t F(U_k(s)) ds$$

Often, one can show that

$U_k \xrightarrow{k \rightarrow \infty} \underline{X}$ on $[-a, a]$, where
 $\underline{X}$ solves IVP.

Ex 3 $x' = x = f(x)$
 $x(0) = x_0$

Then $U_0(t) = x_0$, $U_1(t) = x_0 + \int_0^t \underbrace{f(U_0(s))}_{x_0} ds$

$$= (1+t)x_0$$

$$u_2(t) = x_0 + \int_0^t \underbrace{f(u_1(s))}_{(1+s)x_0} ds = \left(1+t+\frac{t^2}{2}\right)x_0$$

...

$$u_k(t) = \left(\sum_{j=0}^k \frac{t^j}{j!}\right)x_0 \xrightarrow{k \rightarrow \infty} \underbrace{e^t x_0}_{\text{unique sol of IVP}} \quad \forall t \in \mathbb{R},$$

Ex 4 $\underline{x}' = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \underline{x} = A\underline{x} = F(\underline{x})$
 $\underline{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$

Unique sol: Recall

$$\underline{x}(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} \quad c \in \mathbb{R}^2$$

is general solution, and $\underline{x}(0) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$
determines $c = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ uniquely, i.e.

$$\underline{X}(t) = \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix} \text{ is unique sol.}$$

Picard iteration

$$U_0(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$$

$$U_1(t) = \underline{X}_0 + \int_0^t F(U_0(s)) ds$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_0^t A U_0(s) ds$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} 0 \\ -1 \end{bmatrix} ds = \begin{bmatrix} 1 \\ -t \end{bmatrix}$$

$$U_2(t) = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_0^t A U_1(s) ds$$

$$= \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \int_0^t \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} 1 \\ -s \end{bmatrix} ds$$

$$= \begin{bmatrix} 1 - t^2/2 \\ -t \end{bmatrix}$$

$$U_3(t) = \begin{bmatrix} 1 + t^2/2! \\ -t + t^3/3! \end{bmatrix}, \dots$$

and

$$U_k(t) \rightarrow \begin{bmatrix} \cos t \\ -\sin t \end{bmatrix}$$

as $k \rightarrow \infty$.

Ex 6.4 Find general solution to

$$\underline{X}' = \underbrace{\begin{bmatrix} 0 & 1 & 1 & 0 \\ -1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{bmatrix}}_A \underline{X}$$

Sol Writing $\hat{I} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$, we have

$$A = \begin{bmatrix} \hat{I} & I \\ 0 & \hat{I} \end{bmatrix}. \quad \text{Further}$$

$$A^2 = \begin{bmatrix} \hat{I}^2 & 2\hat{I} \\ 0 & \hat{I}^2 \end{bmatrix}, \quad \dots \quad A^j = \begin{bmatrix} \hat{I}^j & j\hat{I}^{j-1} \\ 0 & \hat{I}^j \end{bmatrix}$$

$$\hat{I}^2 = -I, \quad \hat{I}^3 = -\hat{I}, \quad \hat{I}^4 = I, \quad \hat{I}^5 = \hat{I}$$

$$\hat{I}^{2j+1} = (-1)^j \hat{I}, \quad \hat{I}^{2j} = (-1)^j I$$

$$e^{At} = \begin{bmatrix} \sum_{j=0}^{\infty} \frac{t^j \hat{I}^j}{j!} & t \sum_{j=1}^{\infty} \frac{t^{j-1} \hat{I}^{j-1}}{(j-1)!} \\ 0 & \sum_{j=0}^{\infty} \frac{t^j \hat{I}^j}{j!} \end{bmatrix}$$

and

$$\sum_{j=0}^{\infty} \frac{t^j \hat{I}^j}{j!} = \sum_{j=0}^{\infty} \frac{(-1)^j t^{2j}}{(2j)!} \mathbf{I} + \frac{(-1)^j t^{2j+1}}{(2j+1)!} \mathbf{I}$$

$$= \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix} = \hat{A}_t$$

$$\Rightarrow e^{At} = \begin{bmatrix} \hat{A}_t & t \hat{A}_t \\ 0 & \hat{A}_t \end{bmatrix}$$

general solution

$$\underline{X}(t) = e^{At} C$$

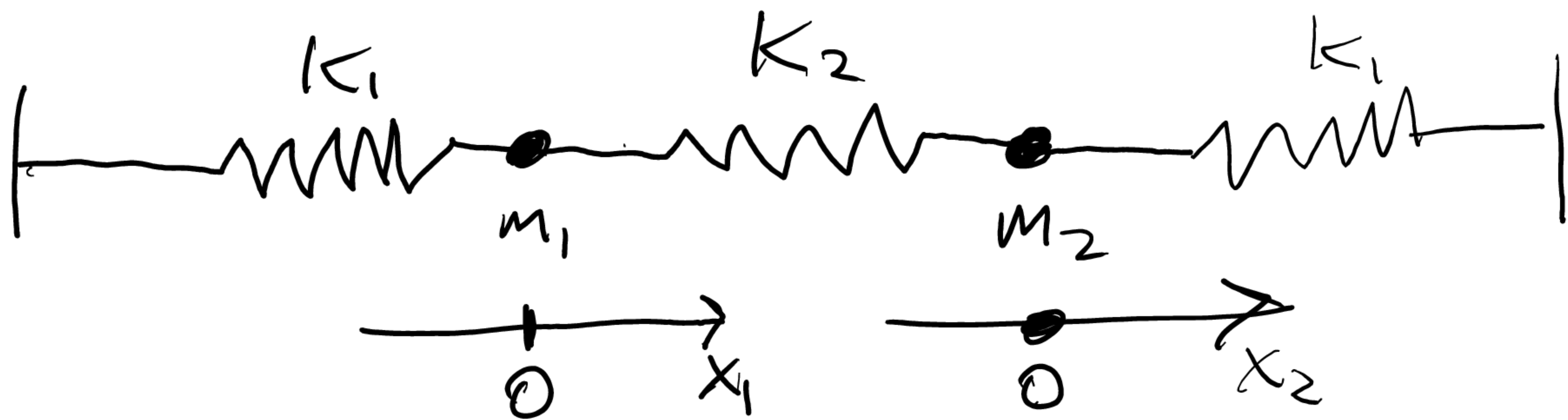
$$= \begin{bmatrix} \hat{A}_t \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} + t \begin{bmatrix} c_3 \\ c_4 \end{bmatrix} \\ \hat{A}_t \begin{bmatrix} c_3 \\ c_4 \end{bmatrix} \end{bmatrix}$$

$$= \begin{bmatrix} (c_1 + c_3 t) \cos t + (c_2 + c_4 t) \sin t \\ -(c_1 + c_3 t) \sin t + (c_2 + c_4 t) \cos t \\ c_3 \cos t + c_4 \sin t \\ -c_3 \sin t + c_4 \cos t \end{bmatrix}$$

6.7, 6.12 a, b, g, j, 6.13, 6.14

6.7 Coupled harmonic oscillators

$$k_1, k_2 > 0, m_1, m_2 > 0$$



resting positions m_1 & m_2

x_i = deviation from resting position m_i

ODE

$$x_1'' = -(k_1 + k_2)x_1 + k_2x_2$$

$$x_2'' = k_2x_1 - (k_1 + k_2)x_2$$

- write ODE as first order system
- Determine eigenvals and eigenvectors.
- Find general solution
- what can be said about periodicity of sol

$$\begin{bmatrix} y_1 = x_1 \\ y_2 = x_2 \\ y_3 = x_1' \\ y_4 = x_2' \end{bmatrix}$$

$$\Rightarrow \begin{aligned} y_1' &= y_3 \\ y_2' &= y_4 \\ y_3' &= -(k_1 + k_2)x_1 + k_2 x_2 \\ &= -(k_1 + k_2)y_1 + k_2 y_2 \end{aligned}$$

$$y_4' = k_2 y_1 - (k_1 + k_2) y_2$$

$$Y' = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ -(k_1 + k_2) & k_2 & 0 & 0 \\ k_2 & -(k_1 + k_2) & 0 & 0 \end{bmatrix} Y$$

$$= \underbrace{\begin{bmatrix} 0 & I_2 \\ C & 0 \end{bmatrix}}_A Y$$

b) char eq. Note $A - \lambda I = \begin{bmatrix} \lambda I_2 & I_2 \\ C & -\lambda I_2 \end{bmatrix}$

and

$$\begin{bmatrix} -\lambda I_2 & I_2 \\ C & -\lambda I_2 \end{bmatrix} \begin{bmatrix} -\lambda I_2 & 0 \\ -C & I_2 \end{bmatrix} = \begin{bmatrix} \lambda^2 I_2 - C & I_2 \\ 0 & -\lambda I_2 \end{bmatrix}$$

$$\begin{aligned} \Rightarrow \det(A) &= \det(\lambda^2 I_2 - C) \\ &= \det \begin{bmatrix} (\lambda^2 + k_1 + k_2) & -k_2 \\ -k_2 & (\lambda^2 + k_1 + k_2) \end{bmatrix} \end{aligned}$$

$$= (\lambda^2 + k_1 + k_2)^2 - k_2^2$$

$$= (\lambda^2 + k_1 + 2k_2)(\lambda^2 + k_1)$$

$E_v \pm i\omega_1$ and $\pm i\omega_2$

where $\omega_1 = \sqrt{k_1}$, $\omega_2 = \sqrt{k_1 + 2k_2}$

$$c) (A - i\omega_1 I) U_1 = 0$$

$$\begin{bmatrix} i\omega_1 & 0 & 1 & 0 \\ 0 & -i\omega_1 & 0 & 1 \\ -(k_1+k_2) & k_2 & -i\omega_1 & 0 \\ k_2 & -(k_1+k_2) & 0 & -i\omega_1 \end{bmatrix} U_1 = 0$$

$$\begin{bmatrix} -i\omega_1 & 0 & 1 & 0 \\ 0 & -i\omega_1 & 0 & 1 \\ -k_2 & k_2 & 0 & 0 \\ k_2 & -k_2 & 0 & 0 \end{bmatrix} U_1 = 0$$

$$U_1 = \begin{bmatrix} i/\omega_1 \\ i/\omega_1 \\ 1 \\ 1 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \\ 1 \\ 1 \end{bmatrix} + i\omega_1^{-1} \underbrace{\begin{bmatrix} 1 \\ 1 \\ 0 \\ 0 \end{bmatrix}}_{V_2}$$

V_1

similarly

$$(A - i\omega_2 I) V_2 = 0 \quad \text{yields}$$

$$V_2 = \underbrace{\begin{bmatrix} 0 \\ 0 \\ 1 \\ -1 \end{bmatrix}}_{V_3} + i\omega_2^{-1} \underbrace{\begin{bmatrix} 1 \\ -1 \\ 0 \\ 0 \end{bmatrix}}_{V_4}$$

Set $P = [V_1 \ V_2 \ V_3 \ V_4]$

$$B = P^{-1} A P = \begin{bmatrix} 0 & \omega_1 & 0 & 0 \\ -\omega_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \omega_2 \\ 0 & 0 & -\omega_2 & 0 \end{bmatrix}$$

$Z' = BZ$ has gen sol

$$Z(t) = \begin{bmatrix} \tilde{C}(\omega_1 t) & 0 \\ 0 & \tilde{C}(\omega_2 t) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$$

where $\tilde{C}(\omega t) = \begin{bmatrix} \cos \omega t & \sin \omega t \\ -\sin \omega t & \cos \omega t \end{bmatrix}$.

$$X(t) = PZ(t) = \underbrace{Z_2 V_2 + Z_4 V_4}_{(x_1, x_2)} + \underbrace{Z_1 V_1 + Z_3 V_3}_{(x_3, x_4)}$$

$$x_1(t) = \omega_1^{-1} (c_1 \cos \omega_1 t + c_2 \sin \omega_1 t) \\ + \omega_2^{-1} (c_3 \cos \omega_2 t + c_4 \sin \omega_2 t)$$

$$x_2(t) = \omega_1^{-1} (c_2 \cos \omega_1 t - c_1 \sin \omega_1 t) \\ + \omega_2^{-1} (c_4 \cos \omega_2 t - c_3 \sin \omega_2 t)$$

d) $\underline{X}(t)$ is a linear combination of

$\frac{2\pi}{\omega_1}$ and $\frac{2\pi}{\omega_2}$ periodic terms,

Suppose $|C_1| + |C_2| \neq 0$ & $|C_3| + |C_4| \neq 0$

(otherwise Problem always has periodic sol)

Then $\underline{X}$ has periodic solutions

if $\omega_2 > \omega_1 > 0$ satisfies $\frac{\omega_2}{\omega_1} \in \mathbb{Q}$

Proof if $\frac{\omega_2}{\omega_1} = \frac{m}{n} \in \mathbb{Q}$, then

$\underline{X}(t)$ is $T = n \cdot \frac{2\pi}{\omega_1}$ periodic, as

$$\begin{aligned}\cos(\omega_2(T+t)) &= (\cos(\omega_2 t + 2m\pi)) \\ &= \cos(\omega_2 t)\end{aligned}$$

$$\begin{aligned}\text{and } \cos(\omega_1(T+t)) &= (\cos(\omega_1 t + 2n\pi)) \\ &= \cos(\omega_1 t) \text{ etc.}\end{aligned}$$

$$\Rightarrow \underline{X}(t+T) = \underline{X}(t)$$

6.12 Compute exponentials of

$$a) A = \begin{bmatrix} 5 & -6 \\ 3 & -4 \end{bmatrix}$$

eigenpairs: $(2, \begin{bmatrix} 2 \\ 1 \end{bmatrix})$, $(-1, \begin{bmatrix} 1 \\ 1 \end{bmatrix})$.

$$P = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}, B = P^{-1}AP = \begin{bmatrix} 2 & 0 \\ 0 & -1 \end{bmatrix}$$

$$e^A = P e^B P^{-1} = P \begin{bmatrix} e^2 & 0 \\ 0 & e^{-1} \end{bmatrix} P^{-1}$$

$$= \begin{bmatrix} 2e^2 & e^{-1} \\ e^2 & e^{-1} \end{bmatrix} \begin{bmatrix} 1 & -1 \\ -1 & 2 \end{bmatrix} = \begin{bmatrix} 2e^2 - e^{-1} & 2(e^{-1} - e^2) \\ e^2 - e^{-1} & 2e^{-1} - e^2 \end{bmatrix}$$

b) $A = \begin{bmatrix} 2 & -1 \\ 1 & 2 \end{bmatrix}$ is on canonical form
form $\begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$ with $\alpha = 2$, $\beta = -1$.

We know then that

1. $\underline{X}' = A\underline{X}$ has general solution

$$\underline{X}(t) = e^{\alpha t} \begin{bmatrix} \cos \beta t & \sin \beta t \\ -\sin \beta t & \cos \beta t \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}$$

but also that it can be written

$$\underline{X}(t) = e^{At} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Taking $t=1$, we get

$$\begin{aligned} e^A &= e^{\alpha} \begin{bmatrix} \cos \beta & \sin \beta \\ -\sin \beta & \cos \beta \end{bmatrix} \\ &= e^2 \begin{bmatrix} \cos 1 & -\sin 1 \\ \sin 1 & \cos 1 \end{bmatrix} \end{aligned}$$

$$g) A = \begin{bmatrix} \lambda & & \\ & \lambda & \\ 0 & & \lambda \end{bmatrix} = \lambda I + N,$$

where $N = \begin{bmatrix} 0 & & \\ & 1 & 0 \\ 0 & & 0 \end{bmatrix}$ is nilpotent of degree $k=3$

Note λI & N commute, so

$$e^A = e^{\lambda I} e^N \text{ and}$$

$$N^2 = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{bmatrix}, \quad N^j = 0 \quad j \geq 3,$$

$$\Rightarrow e^N = N^0 + N^1 + \frac{N^2}{2} =$$

$$= \begin{bmatrix} 1 & & \\ & 1 & \\ 1 & & 1 \end{bmatrix}$$

$$\text{So } e^A = \begin{bmatrix} e^\lambda & & \\ & e^\lambda & \\ & & e^\lambda \end{bmatrix} e^N$$

$$= e^\lambda \begin{bmatrix} 1 & & \\ & 1 & \\ \frac{1}{2} & & 1 \end{bmatrix}.$$

6.13 Find an example of two matrices A, B such that

$$e^{A+B} \neq e^A e^B.$$

Sol $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$

Note $AB = B \neq BA = 0.$

$$A+B = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}, \text{ so } (A+B)^j = A+B \quad \forall j \geq 1.$$

$$\Rightarrow e^{A+B} = I + \sum_{j=1}^{\infty} \frac{1}{j!} \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$$

$$= \begin{bmatrix} e & e-1 \\ 0 & 1 \end{bmatrix}$$

$$\text{Next, } A^j = A \quad \forall j \geq 1 \quad \& \quad B^j = 0 \quad j \geq 2$$

$$\Rightarrow e^A e^B = \begin{bmatrix} e & 0 \\ 0 & 1 \end{bmatrix} \underbrace{(I+B)}_{e^B}$$

$$= \begin{bmatrix} e & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$$

$$= \begin{bmatrix} e & e \\ 0 & 1 \end{bmatrix}.$$

6.14 Show that if $AB=BA$, then

$$a) e^A e^B = e^B e^A$$

$$b) e^A B = B e^A$$

Sol

a) Note $A+B=B+A$ always.

Since AB commutes,

$$e^A e^B = e^{A+B} = e^{B+A} = e^B e^A.$$

b) Let $\Sigma_N = \sum_{j=0}^N \frac{A^j}{j!}$. Then $\Sigma_N \rightarrow e^A$,

and

$$\|\Sigma_N B - e^A B\|_2 \leq \|\Sigma_N - e^A\|_2 \|B\|_2 \rightarrow 0$$

$$\Rightarrow \Sigma_N B \rightarrow e^A B,$$

similarly $B \Sigma_N \rightarrow B e^A$. Further,

$$\Sigma_N B = \sum_{j=0}^N \frac{A^j B}{j!} = \sum_{j=0}^N \frac{B A^j}{j!} = B \Sigma_N$$

$$\Rightarrow e_B^A = \lim_{N \rightarrow \infty} \Sigma_N B = \lim_{N \rightarrow \infty} B \Sigma_N = B e^A.$$